

Transformational Proof Basics - Congruence

Our proofs are based on the definitions, assumptions, and basic theorems listed here.

Basic Transformational Definitions

Transformation of the Plane: A one-to-one function whose domain and range are the entire plane.
Isometry (or rigid motion): A transformation of the plane that preserves distance and angles.
Symmetry of a Figure: An isometry for which the figure, as a whole, is invariant. Individual points in the figure need not be fixed.
Congruence: Two figures are *congruent* if one can be superposed on the other by a sequence of isometries.

Definitions of Reflection, Rotation, Vector, and Translation

Reflection: A reflection in a line b maps any point on b to itself, and any other point P to a point P' so that b is the perpendicular bisector of $\overline{PP'}$.
Rotation: Given a point O and a directed angle θ, the image of a point $P \neq O$ under a rotation with center O and angle θ is a point P' on the circle centered at O with radius OP, such that $\angle POP'=\theta$. The image of O is O. (θ is positive for a counterclockwise rotation, negative for clockwise.) A 180° rotation is also known as a half-turn or reflection in a point.
Vector – Informal Definition for Students: An arrow specifying distance and direction for which position doesn't matter. Two representatives of the same vector will be parallel, have the same length, and point in the same direction. $\vec{v} = \vec{w}$ means they are two representatives of the same vector. (The same notation is used for vector and ray, so you must tell which meant by context.)
Opposite Vectors: Two equal-length vectors pointing in opposite directions. $-\vec{v}$ denotes the vector opposite $\vec{v}$.
Parallel Vectors: Vectors that point in the same or opposite directions. Representatives of parallel vectors are either parallel or collinear.
Vector – Formal Definition for Teachers: This relies on the notion of an equivalence class. The definition, which requires three preliminary definitions, is given at the end of this document.
Translation: Given a vector $\vec{V}$, the image of a point P under a translation by $\vec{V}$, is a point P' such that $\overrightarrow{PP'} = \vec{V}$.

Definitions of Special Triangles and Quadrilaterals

In this approach, we define special triangles and quadrilaterals in terms of their symmetries. We begin each section of *Symmetry Defs and Properties – Triangles and Quads* with a symmetry definition. In *Proving Triangles and Quads Are Special,* we establish conditions that imply a figure has the defining symmetry. In doing so, we will have shown that the traditional definitions are equivalent to the symmetry definitions.

Assumptions

The following five assumptions are sufficient for the mathematically experienced. When working with students or developing curriculum, many of the basic theorems proved below can be added to the set of assumptions because many students will think they are obvious.

1. The parallel postulate: Through a point outside a given line, one and only one line can be drawn parallel to the given line.
2. Two distinct lines meet in at most one point.
3. A circle and a line meet in at most two points.
4. Two distinct circles meet in at most two points.
5. Reflection preserves distance and angle measure.

Theorems proved in *Triangle Congruence and Similarity: A Common-Core-Compatible Approach*, by Henri Picciotto and Lew Douglas

(http://www.mathedpage.org/transformations/triangle-congruence-similarity-v2.pdf)

1. There is a reflection that maps any given point P into any given point Q.
2. A point P is equidistant from two points A and B if and only if it lies on their perpendicular bisector.
3. If two segments AB and CD have equal length, then one is the image of the other with C the image of A and D the image of B, under either one or two reflections.
4. Congruence Criteria for Triangles: SSS, SAS, and ASA.

Basic Theorems

If these theorems are obvious to your students, supplying formal proofs for them will likely be counterproductive. We will, however, supply proofs, or references to proofs, at the end of this list.

1. The composition of two reflections in intersecting lines is a rotation around their point of intersection. The angle of rotation is twice the directed angle between the lines going from the first reflection line to the second (either clockwise or counterclockwise). This implies that any rotation can be decomposed into two reflections.
2. The composition of two reflections in parallel lines is translation. The translation vector is perpendicular to the lines, points from the first line to the second, and has length twice the distance between the lines. This implies that any translation can be decomposed into two reflections.
3. Reflection, rotation, and translation preserve collinearity, betweenness, segment length and angle measure.
4. If A →A' and B →B' under a reflection, rotation, or translation, segment AB must map onto segment A'B'.
5. Reflections, rotations, and translations map lines to lines.
6. If A' = B under a reflection, then B' = A.
7. If A' = B under a half-turn, then B' = A.
8. The composition of translations is commutative.
9. The translation image of a line is either the line itself (if the vector has the same direction as the line) or a line parallel to it (if it doesn't).

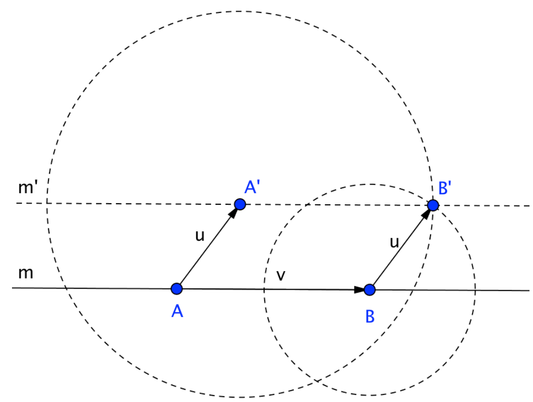
10. Any representative of a vector can be superimposed on any other by a translation.
11. If two lines are cut by a transversal, an angle on one parallel is the translation image of an angle on the other if and only if the lines are parallel.
12. The image of segment AB under a half-turn around its midpoint is the segment BA. That is, A' = B and B' = A.
13. A line is its own image under a half-turn around a point on the line.
14. When two lines intersect, the vertical angles are equal.
15. If two lines are cut by a transversal, they are parallel if and only if the alternate interior angles are equal.
16. The image of a line under a half-turn around a point not on the line is a line parallel to the pre-image.
17. The sum of the angles of a triangle is 180˚. An exterior angle of a triangle is equal to the sum of the remote interior angles, and is therefore greater than either one.
18. The sum of the interior angles of a quadrilateral is 360˚.

Proofs (or References to Proofs) of the Basic Theorems

1. The composition of two reflections in parallel lines is translation. The translation vector is perpendicular to the lines, points from the first line to the second, and has length twice the distance between the lines. This implies that any translation can be decomposed into two reflections.
   Proof Reference:
   http://www.mathedpage.org/transformations/isometries/four/index.html#two
   Theorem 3.
2. The composition of two reflections in intersecting lines is a rotation around their point of intersection. The angle of rotation is twice the directed angle between the lines going from the first reflection line to the second (either clockwise or counterclockwise). This implies that any rotation can be decomposed into two reflections.
   Proof Reference:
   http://www.mathedpage.org/transformations/isometries/four/index.html#two
   Theorem 4.
3. Reflection, rotation, and translation preserve segment length, angle measure. collinearity, and betweenness.
   Proof: We assumed that reflection preserves segment length and angle measure. Since rotation and translation are compositions of two reflections, they preserve them also. If points A, B and C are in order on a line, ∠ABC is a straight angle and B is between A and C. Since angles are preserved by reflection, rotation, and translation, ∠A'B'C' is also a straight angle. Therefore, A', B', and C' are collinear and B' is between A' and C'.
4. If A →A' and B →B' under a reflection, rotation, or translation, segment AB must map onto segment A'B'.
   Proof: Because reflection, rotation, and translation preserve collinearity and betweenness, and because all three are one-to-one, segment AB must map onto segment A'B' under them.

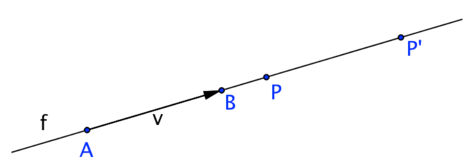
5. Reflections, rotations, and translations map lines onto lines.
   Proof: Because these transformations preserve distance and collinearity, because points arbitrarily far apart can be chosen on the pre-image line, the image of a line must be a line, not a segment.
6. If A' = B under a reflection, then B' = A.
   Proof: If *l* is the reflection line, then *l* is the perpendicular bisector of segment AB and of segment BA.
7. If A' = B under a half-turn, then B' = A.
   Proof: 180˚ rotation clockwise is equivalent to 180˚ rotation counter-clockwise. So, a half-turn is its own inverse.
8. The composition of translations is commutative.

   Notation: $T_u(B)$ is the image of *B* under a translation with vector *u*. $T_u(\overleftrightarrow{AB})$ is the image of line *AB* under a translation with vector *u*, and similarly with segments and vectors.

   Proof: Let *u* and *v* be the two translation vectors. If the vectors are parallel, take representatives on the same line. Mark this line with numbers to make it a number line. The result follows from commutativity of addition.

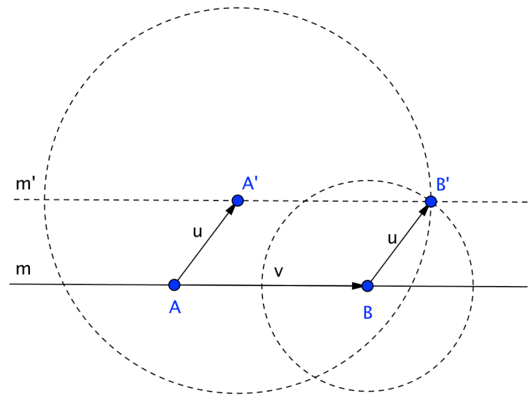
   If their directions are different, let $T_u(A) = A'$, $T_u(B) = B'$ and $T_v(A) = B$. We need to show that $T_v(A') = B'$.
   Let $T_v(A') = C$. By the definition of vector, $A'C = AB$ (the length of *v*).
   $A'B' = AB$ also, since $A'B' = T_u(AB)$ and translation preserves distance. Since $A'B' = A'C$, both B' and C are on a circle centered at A', with radius AB.
   Likewise, $BB' = AA'$ (the length of *u*), and $BC = AA'$ since $BC = T_v(AA')$. Therefore, both B' and C are on a circle centered at B, with radius AA'.
   Thus, B' and C are both at the intersection of the two circles. Let *m* be the line through *A* and *B*. Then $m' = T_u(m)$ is the line through A' and B'. B' must be on the same side of *m* as A', because the other intersection is $T_{-u}(B)$. C must be on the same side of *m* as A' because $T_v(A') = C$ and *v* is the direction of *m*. Therefore B' and C are the same intersection point, so $T_u(T_v(A) = T_u(B) = B'$ and $T_v(T_u(A) = T_v(A') = B'$.
9. The translation image of a line is either the line itself (if the vector has the same direction as the line) or a line parallel to it (if it doesn't).
   Proof:
   Part 1: The translation image of a line is the line itself if the vector has the same direction as the line.

   Proof: Let $\vec{v} = \overrightarrow{AB}$ be the translation vector. If *P* is an arbitrary point on line f, $\overrightarrow{PP'} = \vec{v}$ by the definition of translation. P' lies on f because $\vec{v}$

moves points in the direction of the line. To locate the pre-image of any point on the line under $\vec{v}$, apply $-\vec{v}$ to it. So, the image of the line is the entire line.

<u>Part 2</u>: The translation image of a line is a line parallel to its pre-image if the vector does not have the same direction as the line.

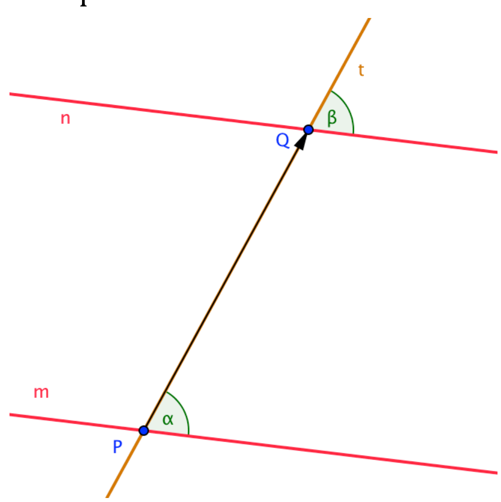


<u>Proof</u>: We can use the same diagram and setup as the previous theorem. Consider a line *m* and a vector *u* in a different direction from *m.* Choose two arbitrary points *A* and *B* on *m*. Let $T_u(A) = A'$, $T_u(B) = B'$, and $\overrightarrow{AB} = v$. We have already shown that $T_v(A') = B'$, and *u* does not have the same direction as v, so the two representations $\overrightarrow{AB}$ and $\overrightarrow{AB}$ of *v* are parallel and not collinear. We have shown that the image of any segment on *m* is a parallel segment on *m'*. Therefore, $m' = T_u(m)$ is parallel to *m*.

10. Any representative of a vector can be superimposed on any other by a translation. <u>Proof</u>: If *u* and *v* are two representatives of the same vector, translate the initial point of *u* to the initial point of *v* by translation T. Since *u* and *v* are either collinear or parallel, since translation maps any line into itself or a parallel line, and since *u* and *v* have the same length and point in the same direction, T(*u*) = *v*.

11. If two lines are cut by a transversal, an angle on one parallel is the translation image of an angle on the other if and only if the lines are parallel.

a) If two parallel lines are cut by a transversal, an angle on one parallel is the translation image of an angle on the other.



<u>Proof</u>: Given parallel lines m and n and transversal t shown in the diagram, translate m by vector $\overrightarrow{PQ}$. m' contains Q and is parallel to m by Theorem 8, Part 2. By the parallel postulate, m' = n. t' = t by Theorem 8 Part 1. Therefore α' = β. Since translation preserves angles, α = β.

b) If two lines are cut by a transversal, and if an angle on one line is the translation image of an angle on the other, then the lines are parallel.

<u>Proof</u>: Suppose angle β is the translation image of angle α, so that α' = β. Because intersection must map to intersection, $\overrightarrow{PQ}$ is the translation vector and t is a transversal because it is common to both angles. Since α' = β, m' = n. By Theorem 8 Part 2, m || n.

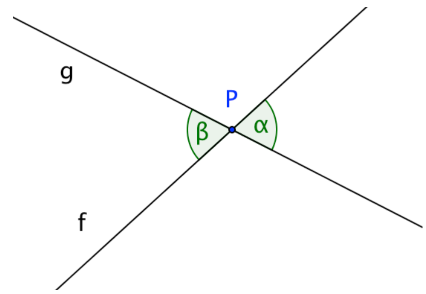
12. The image of segment AB under a half-turn around its midpoint is the segment BA. That is, A' = B and B' = A.

<u>Proof</u>: If M is the midpoint of segment AB, ∠AMB = 180° and MA = MB. By the

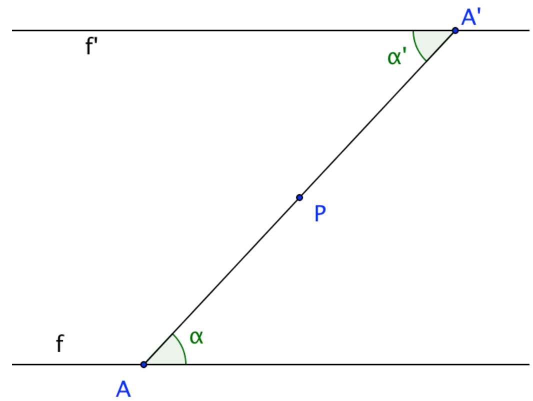
definition of rotation, A' = B and B' = A under a half-turn around M. Since the image of a segment is a segment, AB is invariant under the half-turn.

13. A line is its own image under a half-turn around a point on the line.
Proof: If O is the center of rotation and P is a point on the line, ∠POP' = 180˚ under the half-turn around O. Therefore, P' is also on the line. Under the half-turn, (P')' = P, so every point on the line is the image of another point.

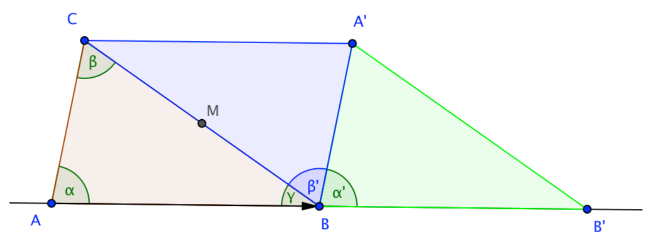
14. When two lines intersect, the vertical angles are equal.
Proof: Rotate angle α 180˚ around P. α' = α because rotation preserves angles. But lines f and g are their own images by Theorem 11. So α' = β and therefore α = β.



15. If two lines are cut by a transversal, they are parallel if and only if the alternate interior angles are equal.
    Proof: An angle alternate interior angle to another is vertical to the translation image of the angle.
16. The image of a line under a half-turn around a point not on the line is a line parallel to the pre-image.
    Proof: Let P be the point not on line f. Pick a point A on f and draw segment AP. Rotate f and AP 180° around P. Since A, P and A' are collinear, AA' is a transversal for lines f and f'. $\alpha' = \alpha$ because rotation preserves angles. Because the alternate interior angles are equal, f' is parallel to f.



17. The sum of the angles of a triangle is 180°. An exterior angle of a triangle is equal to the sum of the remote interior angles, and is therefore greater than either one.
    Proof: Rotate ΔABC 180° around M, the midpoint of segment BC. By Theorem 11, B' = C and C' = B. $\beta' = \beta$ because rotation preserves angles and BA' || AC by Theorem 14. Now translate ΔABC by vector $\overrightarrow{AB}$. A, B and B' are collinear by Theorem 8 Part 1, and C' = A' for two reasons: the image of AC is a line through B parallel to AC by Theorem 8 Part 2, and BA' = AC because translation preserves distance. Because ∠ABB' = 180°, $\alpha' + \beta' + \gamma = \alpha + \beta + \gamma = 180°$. Also from the diagram exterior ∠CBB' = $\alpha + \beta$ and is greater than either α or β. (An informal version of this proof can be discussed with students after asking them to find a tessellation based on a scalene triangle tile.)



18. The sum of the interior angles of a quadrilateral is 360°.
    Proof: Draw a diagonal from the vertex of a reflex angle if there is one, arbitrarily otherwise. The diagonal divides the quadrilateral into two triangles, the sum of whose angles is 180°. The angles of both triangles, together, make up the interior angles of the quadrilateral.

Vector (Formal definition for teachers): Three preliminary definitions are required.

1. Equivalence Relation
   ≡ is an equivalence relation in the set *S* if it is
   a. Reflexive: $x \equiv x$ for all x in S
   b. Symmetric: For all x, y in S, if $x \equiv y$, then $y \equiv x$
   c. Transitive: For all x, y, z in S, if $x \equiv y$ and $y \equiv z$, then $x \equiv z$
2. Equivalence Class
   If *S* has an equivalence relation ≡, then an equivalence class of *S* is a subset *S* consisting of all members of *S* that are equivalent to any one particular element.

3. Directed Line Segment
   A directed line segment is a segment with specified initial and final points. Segment $\overline{AB}$ has two associated directed segments: one with initial point A, the other with initial point B.
4. Vector
   Vector $\overrightarrow{AB}$ is an equivalence class of directed line segments that are equivalent to the one from A to B under the following equivalence relation: Two directed line segments are equivalent if and only if they are parallel and have the same length.