Henri Picciotto

A New Path to the Quadratic Formula

Many students enter ninth grade already familiar with the quadratic formula. Many others learn it in ninth grade. Some can even sing it! Unfortunately, the formula has little meaning for most students. For many, the traditional derivation of the formula by completing the square (**fig. 1**), if it is shown to them, is more baffling than illuminating. As a teacher, I value student understanding, and early on in my career as an algebra teacher, I found this state of affairs disturbing. My first response was to have students complete the square repeatedly, using numbers at first and then the parameters, in the hope that this process would lead to understanding. Alas, over time I realized that for many if not most of my students, additional symbol manipulation did not throw additional light on the subject. I needed to come at this lack of understanding some other way.

Thus was launched an on-and-off quest that led to multiple approaches to this subject and a deeper understanding on my part. (Several student activity sheets about this and closely related material are available for downloading at www.picciotto.org/math-ed/parabolas/.)

This department focuses on mathematics content that appeals to secondary school teachers. It provides a forum that allows classroom teachers to share the mathematics from their work with students, their classroom investigations and projects, and their other experiences. We encourage submissions that pose and solve a novel or interesting mathematics problem, expand on connections among different mathematical topics, present a general method for describing a mathematical notion or solving a class of problems, elaborate on new insights into familiar secondary school mathematics, or leave the reader with a mathematical idea to expand. Send submissions to "Delving Deeper" by accessing **mt.msubmit.net**.

"Delving Deeper" can accept manuscripts in ASCII or Word formats **only**.

*Edited by **Al Cuoco**, acuoco@edc.org*
Center for Mathematics Education, Education Development Center
Newton, MA 02458

***E. Paul Goldenberg**, pgoldenberg@edc.org*
Center for Mathematics Education, Education Development Center
Newton, MA 02458

Because I came to high school mathematics teaching after ten years or so in elementary education, my quest initially led through manipulatives (Picciotto 1995, pp. 130–44). (See **fig. 2**.) This approach is limited to the case where $a = 1$ and $b > 0$, but it generalizes readily to the traditional derivation and can provide a solid foundation for understanding that derivation.

As graphing technology became available, the use of functions as a way to think about solving equations gained currency. In this view, the solution to $ax^2 + bx + c = 0$ is the set of x-intercepts of

If $0 = ax^2 + bx + c$, complete the square:

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

$$-\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$-\frac{c}{a} + \left(\frac{b^2}{4a^2}\right) = \left(x + \frac{b}{2a}\right)^2$$

$$-\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \sqrt{\left(x + \frac{b}{2a}\right)^2}$$

$$\pm\frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$\text{So, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Fig. 1 Derivation of the quadratic formula by completing the square

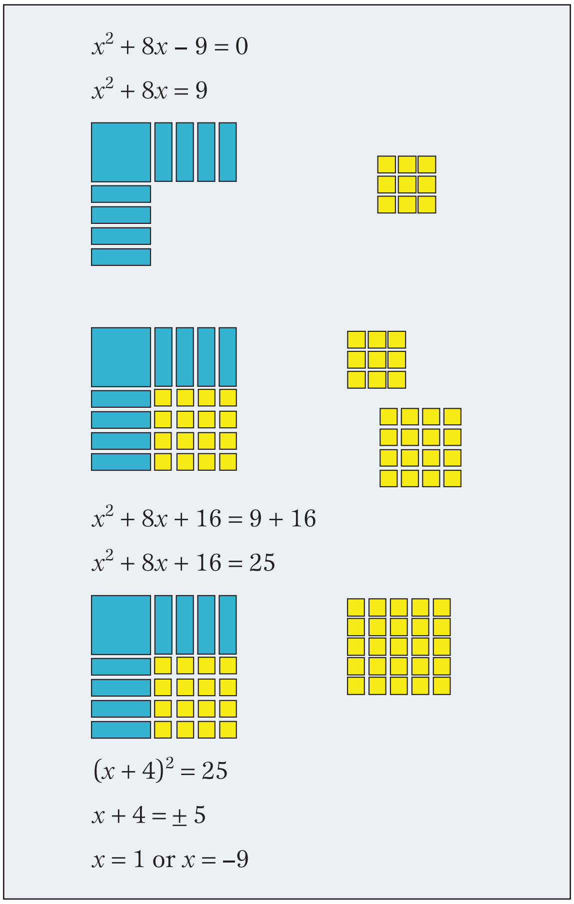

Fig. 2 Manipulatives can be somewhat helpful to students trying to understand how to complete the square

the function $y = ax^2 + bx + c$. I was never convinced that *all* mathematics can be taught by way of functions, but I became intrigued by the possibility of using that approach to derive the quadratic formula, and in fact I did find such a derivation (Wah and Picciotto 1994, pp. 496–98). Here it is:

> If there are roots p and q, the function can be written in factored form:
> $y = a(x - p)(x - q) = ax^2 - a(p + q)x + apq$ (1)

It follows that the product of the roots is c/a, since $c = apq$ and the sum of the roots is $-b/a$, since $b = -a(p + q)$. Let us use this information to find (h, v), the coordinates of the vertex. The average of the roots, h, is $-b/2a$. To find v, we substitute this expression into the formula, simplify, and get

$$v = \frac{-b^2 + 4ac}{4a}.$$

There are real roots if $a > 0$ and $v \le 0$ (a "smiling" parabola with vertex on or below the x-axis) or if $a < 0$ and $v \ge 0$ (a "frowning" parabola with vertex on or above the x-axis). In both cases, there are real roots if $b^2 - 4ac$ is nonnegative; we have rediscovered the discriminant.

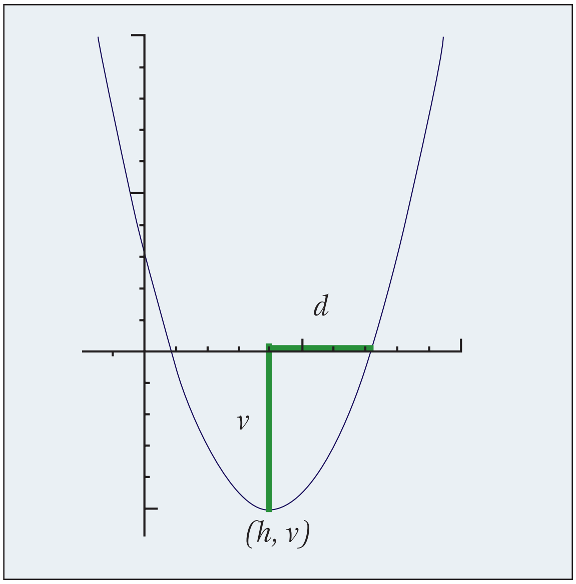

Fig. 3 Graph of a quadratic with two real roots

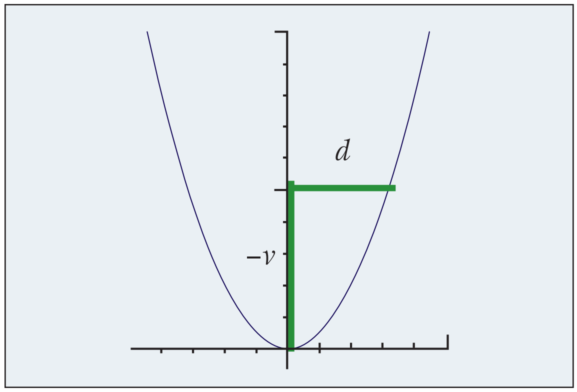

Fig. 4 The quadratic from **figure 3** shifted so that its vertex is at the origin

Let us consider the case where there are real roots and $a > 0$. (The case $a < 0$ works similarly.) The graph looks something like that in **figure 3**. The x-intercepts are on either side of the vertex, at a distance d. So we have

$$x = \frac{-b}{2a} \pm d.$$

To find d, shift the parabola so that its vertex is at the origin (see **fig. 4**). Its equation is now simply $y = ax^2$. The points that were the x-intercepts now have the coordinates $(\pm d, -v)$. It follows that

$$-v = ad^2, \text{ or } \frac{b^2 - 4ac}{4a} = ad^2.$$

Thus,

$$d = \frac{\sqrt{b^2 - 4ac}}{2a}, \text{ and } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In **(a)**, the two intercepts of the parabola are –2 and –8, as can be seen in the factored form of the equation. In **(b)**, a rectangle with area $x^2 + 10x + 16$ is shown in the standard representation by using algebra manipulatives, in this case, *Lab Gear* (Picciotto 1995). The dimensions of the rectangle are $(x + 2)$ and $(x + 8)$. The pieces are one x^2 block, ten x blocks, and sixteen 1 blocks. Because the area of the yellow rectangle is 16 and the total number of *x*s that need to be arranged is 10, it follows that the two numbers we are looking for multiply to 16 and add up to 10. These numbers are 2 and 8, which can also be seen in **(c)**, where the constant product graph of $xy = 16$ and the constant sum graph of $x + y = 10$ intersects at (2, 8) and (8, 2).

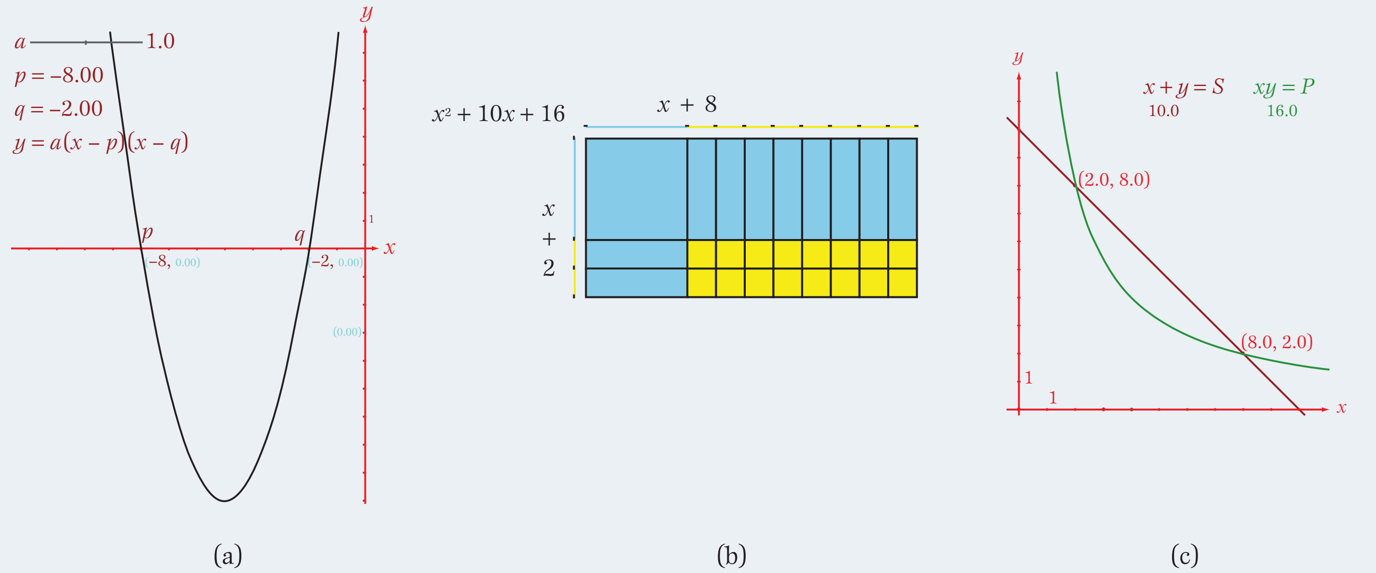

Fig. 5 Three visual representations of $y = (x + 2)(x + 8) = x^2 + 10x + 16$

A NEW PATH: LINE AND HYPERBOLA

Around the same time, I started rethinking first-year algebra with the help of my colleagues, especially Anita Wah. In line with the trend to look at many aspects of mathematics in terms of functions, she suggested that we ask students to make a connection between the intersections of the graphs of constant sums and constant products and the factoring of trinomials (Wah and Picciotto 1994, p. 176.) Over time, this idea turned into a standard assignment (a minireport or poster) for ninth graders at my school and was expanded to include the factored form of quadratic functions and the factoring of trinomials as represented in *Lab Gear* (Picciotto 1995). (See **fig. 5**.) Years later, when I was correcting a student report on these representations, it occurred to me that there must be a way to derive the quadratic formula starting with the line and the hyperbola. Indeed, there is.

As we saw, equation (1) on the previous page yields an expression for the sum of the roots ($p + q = –b/a$). It also gives us an expression for the product of the roots: $pq = c/a$ since $c = apq$. The converse of this is that if we have two numbers p and q whose sum is $–b/a$ and whose product is c/a (with $a \neq 0$), then p and q are the solutions to the quadratic equation $ax^2 + bx + c = 0$. That the converse is true can be shown as follows:

$$\begin{cases} p + q = \dfrac{-b}{a} \\ pq = \dfrac{c}{a} \end{cases} \qquad (2)$$

$$\begin{cases} ap + aq = -b \\ apq = c \end{cases} \qquad (3)$$

$$\begin{cases} apq + aq^2 = -bq \\ apq = c \end{cases} \qquad (4)$$

Therefore, $aq^2 + bq + c = 0$, which is what we wanted to show. [**Editor's note**: Don't worry. We didn't lose a root. In step (4), we could as easily have multiplied each term by p instead of q. In that case, we would arrive at $ap^2 + bp + c = 0$.] In other words, solving the first system of equations above is equivalent to solving the quadratic equation $ax^2 + bx + c = 0$.

If $c = 0$, the equation can readily be solved by factoring, and it is easy to check that the solution satisfies the quadratic formula. If $c \neq 0$, we will solve the system in (2) above with the help of a graph. To make the process clearer, we will use the equivalent equations $x + y = b/a$ and $xy = c/a$.

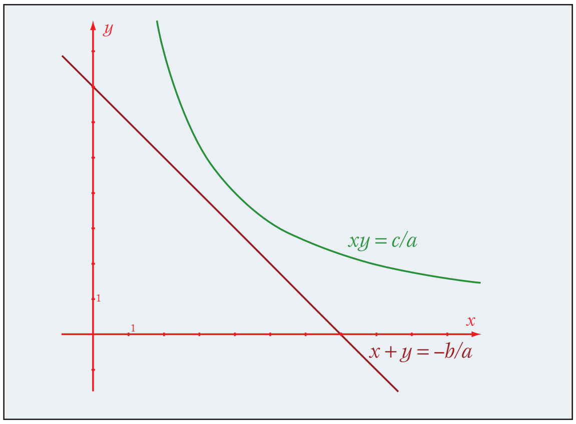

Fig. 6 The system may have no solutions in case 1.

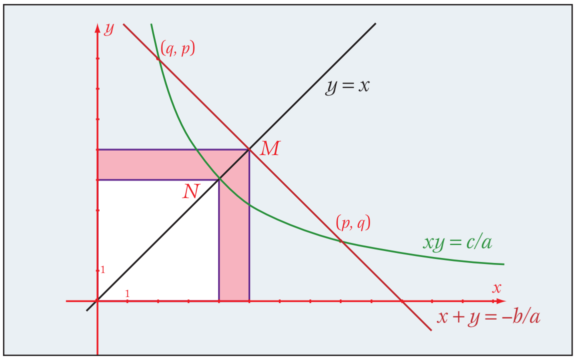

Fig. 7 Case 1 may have a solution.

Case 1: c/a > 0

In this case, it is possible to have no solution to this system (see **fig. 6**). If there is a solution, we have the situation illustrated in **figure 7**. Consider the square with opposite vertices at the origin and M and the square with opposite vertices at the origin and N. A solution to our system exists if the difference of their areas (shaded in the figure) is nonnegative. M has coordinates

$$\left(\frac{-b}{2a}, \frac{-b}{2a}\right),$$

because its coordinates satisfy

$$\begin{cases} y = x \\ x + y = \dfrac{-b}{a}. \end{cases}$$

N has coordinates

$$\left(\sqrt{\frac{c}{a}}, \sqrt{\frac{c}{a}}\right),$$

because its coordinates satisfy

$$\begin{cases} y = x \\ xy = \dfrac{c}{a}. \end{cases}$$

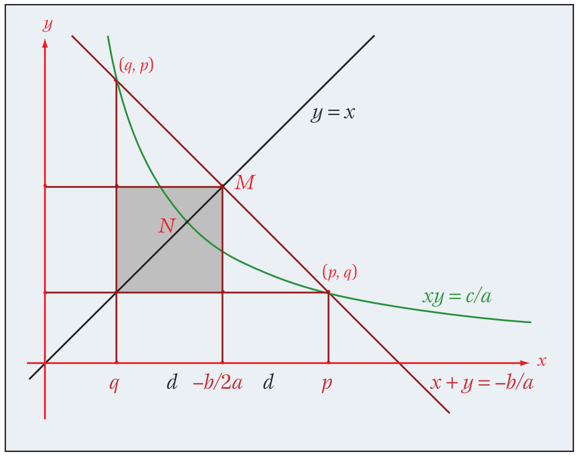

Fig. 8 Determining *d* will provide a solution to the system.

It follows that the shaded area is

$$\frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Because the denominator is always positive, the fraction is nonnegative when its numerator, $b^2 - 4ac$, is nonnegative. We have re-rediscovered the discriminant!

To solve this system, we need to find p and q in terms of a, b, and c. Because $-b/a$ is the sum of $p + q$, $-b/2a$ is the average. In other words, for some number d, we have

$$p = \frac{-b}{2a} + d \text{ and } q = \frac{-b}{2a} - d.$$

More compactly, we have

$$x = \frac{-b}{2a} \pm d,$$

and we have solved this system if we can find d.

To do that, consider **figure 8**, where d is the side of the shaded square. That square has an area equal to the area of the larger square (the one with opposite vertices at the origin and M) minus its unshaded portion. It is not hard to see that this unshaded area equals the area of the rectangle with opposite vertices at the origin and (p, q). But because the hyperbola (p, q) is the locus of points whose coordinates have a constant product, this rectangle has the same area as the square with opposite vertices at the origin and N. Therefore, the shaded square has the same area as the shaded polygon of **figure 6**, an area that we have already calculated. We conclude that

$$d^2 = \frac{b^2 - 4ac}{4a^2},$$

$$d = \frac{\sqrt{b^2 - 4ac}}{2a},$$

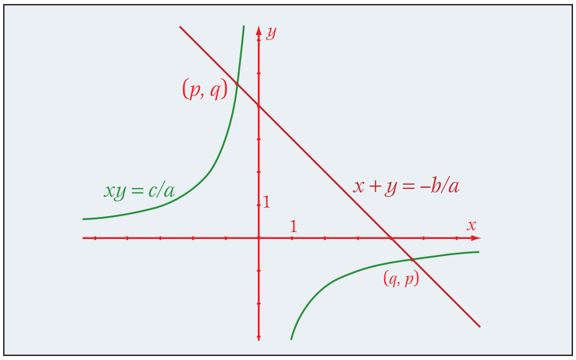

Fig. 9 Case 2 will always provide a solution.

and, therefore, that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as expected.

Case 2: c/a < 0

In this case, a solution always exists (as can be seen in **fig. 9**). To find the solution, once again note that

$$x = \frac{-b}{2a} \pm d,$$

with d the side of the shaded square (see **fig. 10**). To find the area of the square, note that the rectangular part below the x-axis is congruent to the rectangle immediately to its right, which in turn is congruent by symmetry across the $y = x$ line to the rectangle sitting atop the shaded square. So the area of the square equals the area of its part in the first quadrant plus the area of the rectangle having opposite vertices at the origin and (q, p). But that rectangle has an area equal to that of the square with opposite vertices at N and the origin (**fig. 11**). So

$$d^2 = \frac{b^2}{4a^2} + \frac{-c}{a},$$

and the derivation proceeds as above.

FURTHER EXPLORATIONS

What makes our profession endlessly fascinating is the interplay between our own exploration of mathematics and that of our students. This case is emblematic: I was motivated to seek nontraditional derivations because of the challenge of teaching for understanding. I enjoyed the search for them, deepened my understanding, and was rewarded with expanded options on how to present a core part of the high school mathematics curriculum.

My challenge to readers of this column: Can you use a similar approach in three dimensions to derive Cardano's formula for the solution of cubic equations?

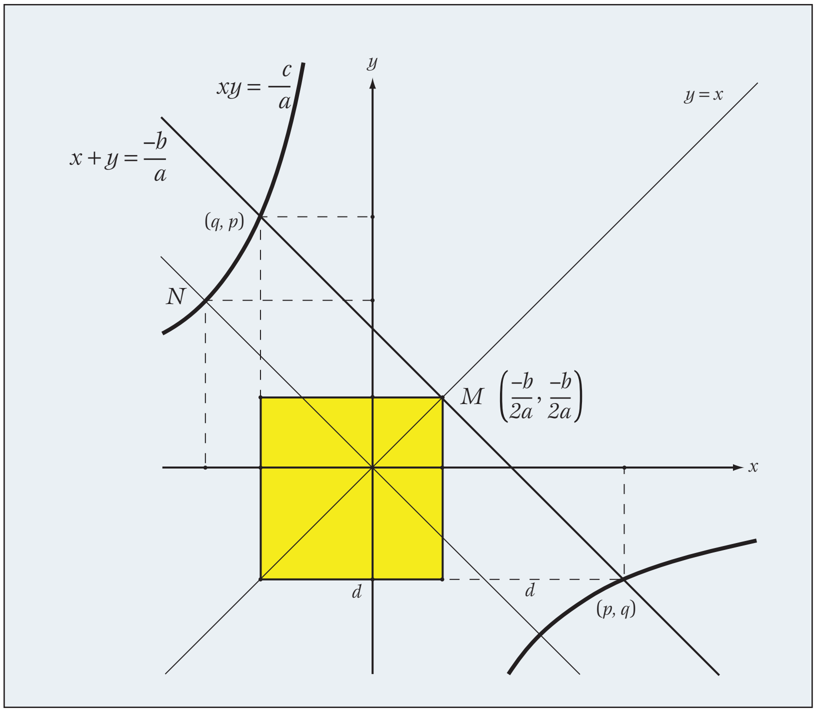

Fig. 10 As before, d is the side of the shaded square.

The area of the shaded square to the left of the y-axis is equal to the area of the rectangle with opposite vertices at the origin and (q, p), since both have area equal to c/a. Consider the upper part of the rectangle, with opposite vertices at (q, p) and $(0, -b/a)$. It is congruent to its mirror image across the $y = x$ line, which is in turn congruent to the shaded rectangle below the x-axis in **figure 10**. Therefore, the shaded areas in **figures 10** and **11** are equal.

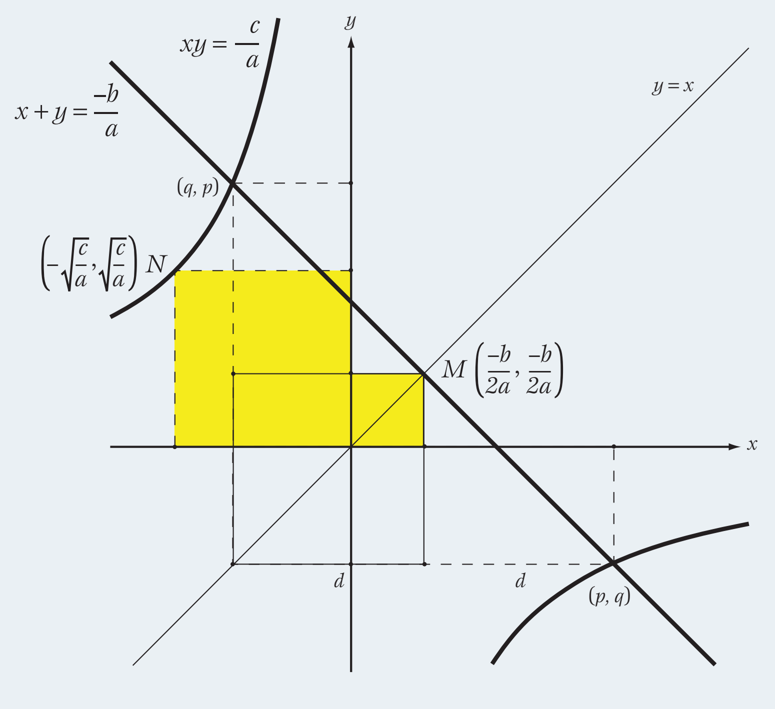

Fig. 11 Determining the solution when $c/a < 0$

Editor's notes: We especially enjoy examining something very familiar from an entirely new perspective. Here, to change perspective, the author switches variables. The traditional derivation of the quadratic formula solves the equation $ax^2 + bx + c = 0$ (in *one* variable) by showing how x depends on (i.e., is derived from) the values of a, b, and c. If we think of the function $f(x) = ax^2 + bx + c$, then those same values of x are the zeros of the function. Graphing $y = f(x)$, we see these as the x-intercepts. But Picciotto presents yet another way to look at the solutions to $ax^2 + bx + c = 0$. Using p and q to name the two values of x that are the solutions, Picciotto points out that p and q must satisfy two constraints: one that describes their sum $p + q$ and one that describes their product pq. These two constraints express q as a function of p in two different ways: as a system of simultaneous equations, only one of which is linear. Where the graphs intersect, both constraints are met, and so the coordinates of either intersection are the solutions to the original equation, $ax^2 + bx + c = 0$. Picciotto shows how familiar elements of the quadratic formula can be derived from certain features of this graph. The change of perspective shows how the two roots of the quadratic relate *to each other.*

Picciotto challenges the reader to extend the idea presented here into three dimensions. This is a challenge also posed by Marion Walter in a submission to this department that we hope to publish in the future. Her challenge involves elevating a familiar plane geometry theorem concerning the intersection of chords of a circle to three dimensions. Our invitation to readers is to explore other examples of moving a familiar idea in high school mathematics from two dimensions to three and submit the results to this department.

REFERENCES

Picciotto, Henri. *Lab Gear Activities for Algebra 1.* New York: Creative Publications, 1995.

Wah, Anita, and Henri Picciotto. *Algebra: Themes, Tools, and Concepts.* New York: Creative Publications, 1994. www.picciotto.org/math-ed/attc/. ∞

HENRI PICCIOTTO, math-ed@picciotto.org, chairs the mathematics department at the Urban School of San Francisco. A curriculum developer and consultant as well as a teacher, he shares ideas on his Web site: www.picciotto.org/math-ed.