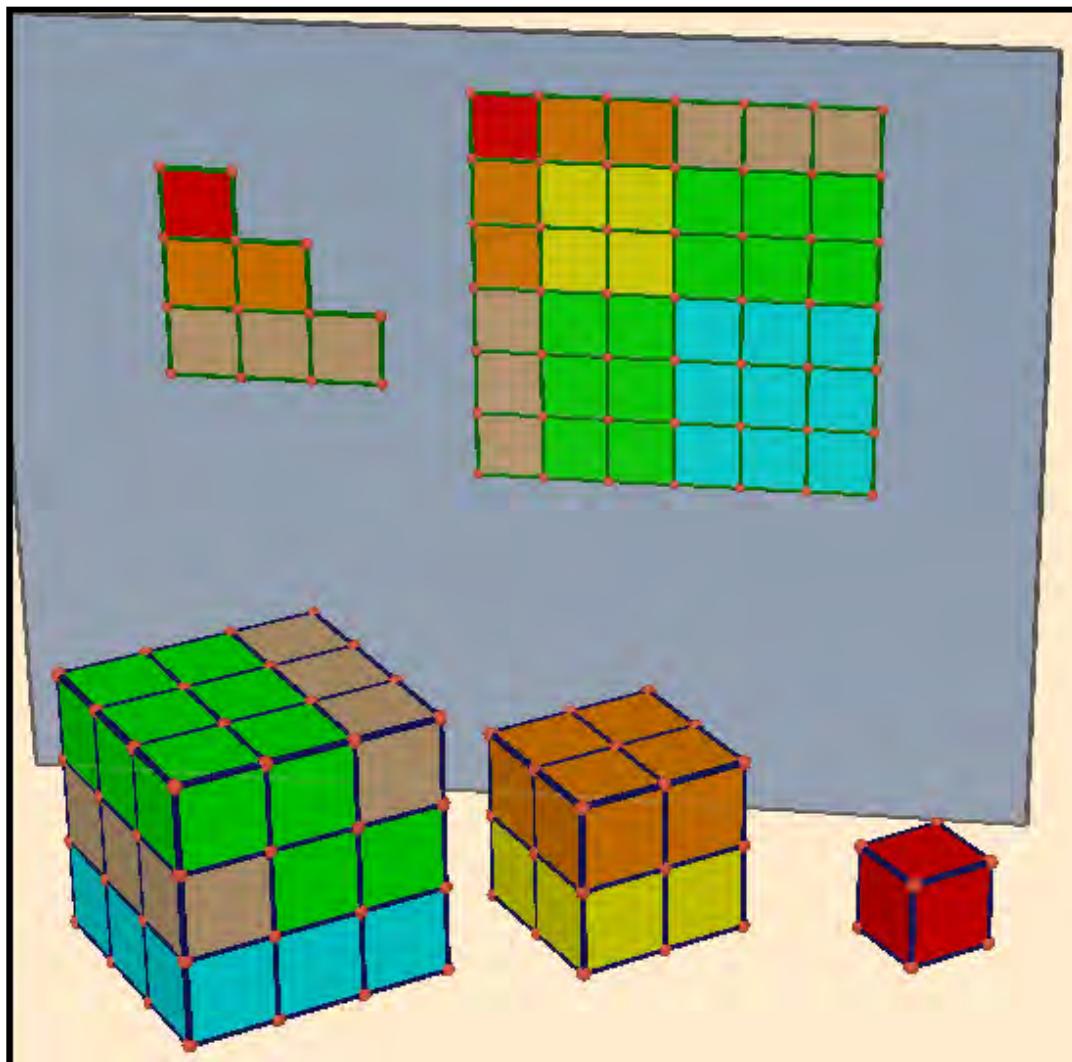


# Selected Integer Sequences

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Slime Numbers: [A152242](#), [A166504](#) ..... p. 2

McNuggets Numbers: [A214777](#) ..... p. 3

Polyomino Perimeter: [A027709](#) ..... p. 6

Staircases (trapezoidal numbers): [A069283](#) ..... p. 9

Figurate Numbers: [A000217](#), [A005891](#), [A000537](#), etc. ..... p. 10

## Slumber Theory

Slumber theory is a silly branch of mathematics, which exists only on this page.

Any number can be **sliced** into a sequence of numbers.

**Example:** 365 can be sliced in four different ways:  $3 \mid 6 \mid 5$ ;  $36 \mid 5$ ;  $3 \mid 65$ ; or  $365$ .

(Note that the slices are indicated by a vertical slash. Note also that in slumber theory, not slicing is considered a form of slicing.)

1. How many ways are there to slice a four-digit number?

A number is **slime** if it can be sliced into a sequence of primes.

**Examples:** 5 is slime, since it is already prime.

2027 is slime ( $2 \mid 02 \mid 7$ )

4,155,243,311 is slime ( $4 \mid 1 \mid 5 \mid 5 \mid 2 \mid 43 \mid 3 \mid 11$ )

2. Which one of the following numbers is slime? 12; 345; 6789
3. 2 is the only even prime. Find the first three even slimes.
4. There are no prime squares. Find the first two slime squares.
5. There are no prime cubes. Find the first two slime cubes.
6. 2 and 3 are the only consecutive numbers that are both prime. Find the first three pairs of consecutive numbers that are both slime.
7. There is no triple of consecutive numbers that are all prime. Find the first two triples of consecutive numbers that are all slime.
8. Prove that there are an infinite number of slime numbers.
9. Find the smallest number that is slime in more than one way. (In other words, it can be sliced into two different sequences of primes.)
10. Find the smallest number that is slime in more than two ways.

A number is a **super-slime** if you get a sequence of primes no matter how you slice it.

**Example:** 53 is a super-slime since 53 and  $5 \mid 3$  are both sequences of primes.

11. Prove that there are only a finite number of super-slimes.

## The McNuggets Problem



One day you go to McDonalds and try to order 13 McNuggets. You are told that is impossible. One can only order 6, 9, or 20 nuggets. You could get 12 (because that's  $6+6$ ) or 15 (because that's  $6+9$ ), but not 13.

What numbers of McNuggets is it possible to order? What numbers of McNuggets are impossible to order? (Assume you must include all the McNuggets you order in the count)

**McNuggets Chart**

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 1  | 2  | 3  | 4  | 5  | 6  |
| 7  | 8  | 9  | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 |
| 37 | 38 | 39 | 40 | 41 | 42 |
| 43 | 44 | 45 | 46 | 47 | 48 |
| 49 | 50 | 51 | 52 | 53 | 54 |
| 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 |
| 67 | 68 | 69 | 70 | 71 | 72 |
| 73 | 74 | 75 | 76 | 77 | 78 |

## McNuggets Extension

Recall *The McNuggets Problem*. As a follow-up, let's generalize it. To make this manageable, we will start with two numbers, not three.

1. You have an unlimited supply of dimes (10 cents) and quarters (25 cents). What amounts can be obtained, and what amounts cannot be obtained by combining them?
2. At Fred's Kitchen Supply, cabinets are available in two lengths: 3 feet and 5 feet. By putting cabinets end to end, Fred can accommodate walls of different lengths. Imagining that kitchens can be arbitrarily large, what length walls are possible to line exactly with cabinets? What lengths are impossible?
3. What numbers can be obtained by adding the numbers 6 and 9 as many times as you want? What numbers cannot be obtained?
4. In 1958, it cost 4 cents to mail a letter in the United States. In 1963, it cost 5 cents. Imagine you have an unlimited supply of 4 and 5-cent stamps. What amounts can you make? What is the largest amount you cannot make?

### Generalizing

5. By now, you are probably aware that some pairs of building block numbers work better than others.
  - a. Which types of pairs allow us to build every number beyond a certain point?
  - b. For the other pairs, what do they allow us to build?
6. Let us use  $p$  and  $q$  as our numbers, with  $p < q$ . To find the largest impossible number, use the strategy we used for the McNuggets.
 

Organize numbers in  $p$  columns.

Circle  $p$  and all the numbers below it.

Circle  $q$  and all the numbers below it.

Circle  $2q$  and all the numbers below it.

Circle  $3q$  and all the numbers below it.

...

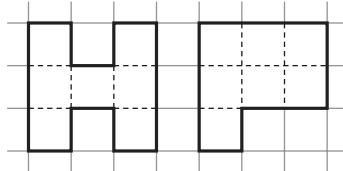
Continue until you have reached all the columns.
7. Explain why this is true:
  - a. The last multiple of  $q$  you circled is  $(p - 1)q$ .
  - b. The last uncircled number is  $(p - 1)q - p = pq - (p + q)$ .

**LAB 8.1**

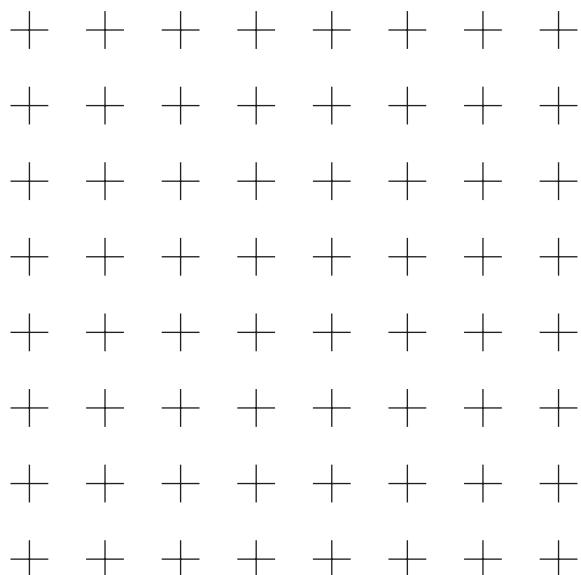
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**Polyomino Perimeter and Area****■ Equipment:** 1-Centimeter Grid Paper, interlocking cubes

A polyomino is a graph paper figure whose outline follows the graph paper lines and never crosses itself. We will only consider polyominoes with no holes, such as these:



1. What are the area and perimeter of each polyomino shown above?
  
  
  
2. Find a polyomino with the same area as the ones above, but with a different perimeter. Sketch it in the grid below.



**LAB 8.1****Polyomino Perimeter and Area (continued)**

Name(s) \_\_\_\_\_

3. Experiment on graph paper or with the help of your interlocking cubes and fill out the following table.

| <b>Area</b> | <b>Minimum perimeter</b> | <b>Maximum perimeter</b> |
|-------------|--------------------------|--------------------------|
| 1           |                          |                          |
| 2           |                          |                          |
| 3           | 8                        | 8                        |
| 4           |                          |                          |
| 5           | 10                       | 12                       |
| 6           |                          |                          |
| 7           |                          |                          |
| 8           |                          |                          |
| 9           |                          |                          |
| 10          |                          |                          |
| 11          |                          |                          |
| 12          |                          |                          |
| 13          |                          |                          |

| <b>Area</b> | <b>Minimum perimeter</b> | <b>Maximum perimeter</b> |
|-------------|--------------------------|--------------------------|
| 14          |                          |                          |
| 15          |                          |                          |
| 16          |                          |                          |
| 17          |                          |                          |
| 18          |                          |                          |
| 19          |                          |                          |
| 20          |                          |                          |
| 21          |                          |                          |
| 22          |                          |                          |
| 23          |                          |                          |
| 24          |                          |                          |
| 25          |                          |                          |
| 26          |                          |                          |

4. Find a formula for the maximum perimeter,  $P_{\max}$ , for a given area  $A$ .
5. Describe a pattern for the minimum perimeter.
6. What would the minimum and maximum perimeters be for the following areas?
- 49 min. \_\_\_\_\_ max. \_\_\_\_\_
  - 45 min. \_\_\_\_\_ max. \_\_\_\_\_
  - 50 min. \_\_\_\_\_ max. \_\_\_\_\_
  - 56 min. \_\_\_\_\_ max. \_\_\_\_\_

**LAB 8.1**

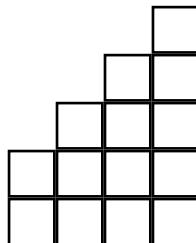
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**Polyomino Perimeter and Area (continued)****Discussion**

- A.** Is it possible to draw a polyomino with an odd perimeter? Explain how to do it, or why it is impossible.
- B.** While filling out the table, what was your strategy for finding the maximum and minimum perimeters?
- C.** Graph minimum and maximum polyomino perimeters with area on the  $x$ -axis and perimeter on the  $y$ -axis. What does the graph show about the formula and pattern you found in Problems 4 and 5?
- D.** Find the polyominoes whose area and perimeter are numerically equal.
- E.** Explain your strategy for answering the questions in Problem 6.

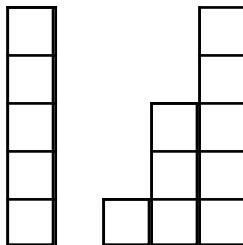
## Staircases

Here is an example of a kind of arrangement of square tiles that we'll call a *staircase*. It has 4 steps and the first step is of height 2.



**Definition:** For this lesson, we will define a staircase as a sequence of stacks of tiles in which each stack is one tile higher than the previous stack. *There must be two or more steps in the staircase* and the first step can be of any height.

These are not staircases:



1. How many tiles would you need to build each of these staircases:
  - a. First step: 7. Number of steps: 8
  - b. First step: 8. Number of steps: 7
  - c. First step: 6. Number of steps: 9
  
2. There are two different 9-tile staircases:  $2 + 3 + 4$  and  $4 + 5$ .
  - a. Find three different 15-tile staircases.
  - b. Find four different 105-tile staircases.
  
3. **Exploration.** Find every possible staircase with each number of tiles from 2 to 34. Hints:
  - Work with other students.
  - Keep organized records of your work.
  - It is not necessary to draw the staircases.
  - Look for strategies: what numbers can be made into two-step staircases? three-step?
  - Look for patterns: what numbers are easiest? what numbers are impossible?
  - What is the average (mean) of a string of consecutive numbers?
  - How is the average related to the sum?
  - What is a good shortcut for adding consecutive numbers?

In general, how can you predict the number of ways a number can be written as a sum of consecutive numbers?

## Introduction to Mathematical Induction

*Mathematical induction* is used to prove that a fact is true for *all* (natural number) values of n.

- ◊ Proving the fact works for the initial term (or terms) of the sequence is *the anchor*.
- ◊ Proving that *if it works for  $n-1$ , then it works for  $n$*  is the *inductive* (or recursive) step.
- ◊ If you prove both, you are done.

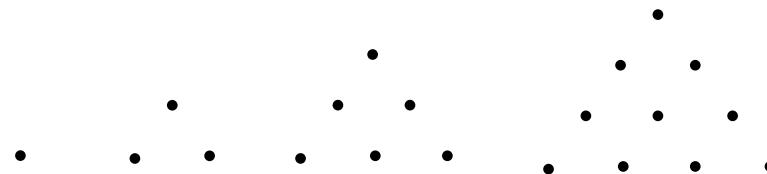
One way to help you prove a formula by mathematical induction uses the understanding you already have of recursive and explicit formulas:

*Recursive Formula:* equation expressing  $x_n$  in terms of  $x_{n-1}$  (and sometimes  $x_{n-2}$ , and so on.)

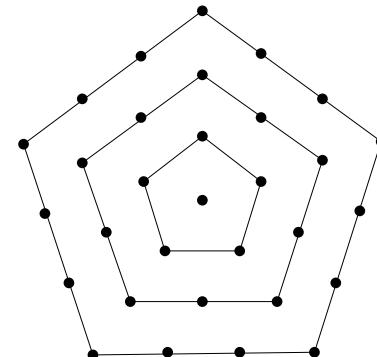
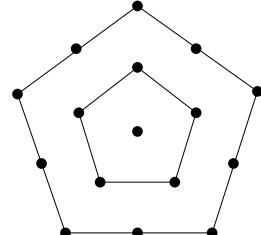
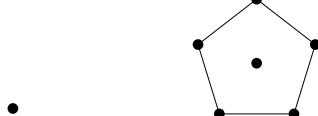
*Explicit Formula:* equation expressing  $x_n$  in terms of n.

For each problem below:

- a. List the first eight terms of the sequence.
- b. Make a conjecture about the explicit formula for  $x_n$ .
- c. Write the same formula for  $x_{n-1}$ .
- d. Find the recursive formula.
- e. Use algebra to derive the equation you found in (b) from the ones you wrote in (c) and (d).
- f. You are now ready to write a proof by mathematical induction! Do it.



1. The  $n^{\text{th}}$  triangular number.
2. The  $n^{\text{th}}$  pentagonal number .



3. The  $n^{\text{th}}$  odd number.
4. The sum of the first n odd numbers.
5. 
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n\cdot(n+1)}$$
6. 
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$$
7. The sum of the first n square numbers.
8. The sum of the first n triangular numbers.
9. The sum of the first n cubes.
10. The number of rectangles of any size (including squares) in an n-by-n checkerboard.