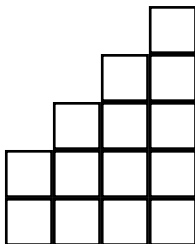


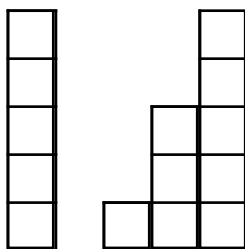
## Staircases

Here is an example of a kind of arrangement of square tiles that we'll call a *staircase*. It has 4 steps and the first step is of height 2.



**Definition:** For this lesson, we will define a staircase as a sequence of stacks of tiles in which each stack is one tile higher than the previous stack. *There must be two or more steps in the staircase and the first step can be of any height.*

These are not staircases:



- How many tiles would you need to build each of these staircases:
  - First step: 7. Number of steps: 8
  - First step: 8. Number of steps: 7
  - First step: 6. Number of steps: 9
- There are two different 9-tile staircases:  $2 + 3 + 4$  and  $4 + 5$ .
  - Find three different 15-tile staircases.
  - Find four different 105-tile staircases.
- Exploration.** Find every possible staircase with each number of tiles from 2 to 34. Hints:
  - Work with other students.
  - Keep organized records of your work.
  - It is not necessary to draw the staircases.
  - Look for strategies: what numbers can be made into two-step staircases? three-step?
  - Look for patterns: what numbers are easiest? what numbers are impossible?
  - What is the average (mean) of a string of consecutive numbers?
  - How is the average related to the sum?
  - What is a good shortcut for adding consecutive numbers?

In general, how can you predict the number of ways a number can be written as a sum of consecutive numbers?

## Staircases Notes

See an extension and more teacher notes in *Algebra: Themes, Tools, Concepts*, 5.9.

**Algebra 1:** By listing 2-step sums, 3-step sums, etc, we get lists of solutions that give us a full solution to the problem, given enough time. Students can be asked to find a formula for each list. The formulas are all linear, with the slope equal to the number of stairs, and intercept equal to a triangular number. (For an  $n$ -stair staircase, use  $T_{n-1}$ .)

**Algebra 2:** This lesson is about arithmetic series of natural numbers with common difference 1. The rest of the lesson in *ATTC* starts an exploration of this.

Note that the sum can be computed easily if one knows the middle term or terms, and the number of terms. For an odd number of terms,  $2k+1$ , with middle term  $m$ , the average value of the terms is  $m$ , and so the sum is  $m(2k+1)$ . For an even number of terms,  $2m$ , with middle terms  $k$  and  $k+1$ , the average of the terms is  $k+1/2$ , and so the sum is  $2m(k+1/2)=m(2k+1)$ . This observation leads to:

**Theorem:** A natural number can be written as a sum of consecutive natural numbers in as many ways as it has odd factors greater than 1.

**Proof:** Let  $n$  be the number.

First we show that for each  $k>0$  such that  $(2k+1)m=n$ , there is a sum of consecutive natural numbers that adds up to  $n$ . Two cases:

- If  $m \geq k+1$ , then the sum has  $2k+1$  terms, and the middle term is  $m$ , with  $k$  terms before, and  $k$  terms after. Since the average term is  $m$ , the sum is  $(2k+1)m=n$ . (Note that if  $m=k+1$ , we have a triangular number.) Since  $m \geq k+1$ , the series' first term is positive.
- If  $m < k+1$ , a series constructed as above would have to start with a negative number, and thus would not be acceptable. Instead, we create a series with middle terms  $k$  and  $k+1$ ,  $m-1$  terms before, and  $m-1$  terms after. Since the average term is  $k+1/2$ , the sum is  $2m(k+1/2)=m(2k+1)=n$

All sequences are different, since their middle term(s) are different.

Now we show that for each sequence of consecutive numbers that adds up to  $n$ , there is an odd factor of  $n$ .

- If the sequence has  $2k+1$  terms, and middle term  $m$ , then  $n=(2k+1)m$ , and  $2k+1$  is our odd factor.
- If the sequence has  $2m$  terms, with middle terms  $k$  and  $k+1$ , then  $n=2m(k+1/2)=m(2k+1)$ , and  $2k+1$  is our odd factor.

Since each odd factor  $>1$  yields a distinct sequence, and each sequence yields an odd factor  $>1$ , there are as many sequences as odd factors  $>1$ .

**Corollary:** Powers of 2 cannot be written as sums of consecutive natural numbers.

**Proof:** They have no odd factors.