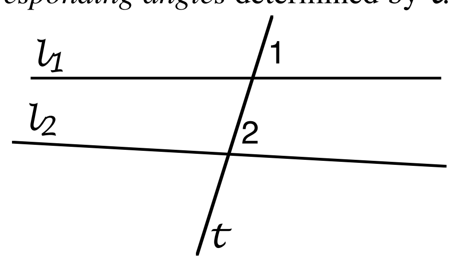
The Geometry of Linear Graphs

As you know, the graph of y = mx + b is a (straight) line, and the equation of a non-vertical line is of the form y = mx + b. More formally:

Theorem: For every pair of parameters m and b, there is a unique non-vertical line such that a point is on the line if and only if its coordinates satisfy y = mx + b. Conversely, for every non-vertical line, there is a unique pair of parameters m and b such that a point is on the line if and only if its coordinates satisfy y = mx + b.

We will prove the theorem, using basic high school geometry, and the knowledge that a point's coordinates are determined by dropping perpendiculars to the axes. Here are the definitions and facts we will need:

Definitions: If a line t intersects two other lines l_1 and l_2, it is called a *transversal*. In this figure, angles 1 and 2 are *corresponding angles* determined by t.



Fact T: If the corresponding angles determined by a transversal are equal, then the two lines are parallel. Conversely, if two lines are parallel, then corresponding angles are equal.

Definition: A quadrilateral whose opposite sides are parallel is called a *parallelogram*.

Fact P: Opposite sides of a parallelogram are equal.

Fact PE: In a quadrilateral, if two opposite sides are both parallel and equal, the quadrilateral is a parallelogram.

Definition: Two polygons are *similar* whenever corresponding sides are proportional and corresponding angles are equal.

Fact AA: If in two triangles two pairs of corresponding angles are equal, then the triangles are similar.

Fact LL: If in two right triangles the legs are proportional, then the triangles are similar.

The heart of the proof lies in these two results:

◊ **Result 1:** For a given m and b, all points (x,y) whose coordinates satisfy y = mx + b belong to a unique non-vertical line.

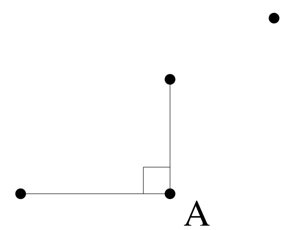
◊ **Result 2:** For a given non-vertical line, there is a unique pair of parameters m and b such that all the points on the line satisfy the equation y = mx + b

We will prove these first, then put them together to prove the theorem.

Result 1: For a given m and b, all points whose coordinates satisfy y=mx+b belong to a unique non-vertical line

We will prove this in two cases: first if $m \neq 0$, then if $m = 0$

If $m \neq 0$, we will start with two different arbitrary values for x, x_Q and x_R, with $x_Q<x_R$. Let $y_Q = mx_Q + b$, and $y_R = mx_R + b$. This gives us two points on the graph of $y = mx + b$: $Q(x_Q, y_Q)$, and $R(x_R, y_R)$. What we would like to show is that a generic point $P(x, y)$ where $y = mx + b$ must be on the line QR.



1. Say that $x < x_Q$.
   a. Label the three unlabeled points on the figure above as P, Q, and R, from left to right. Connect PQ and QR. You will get a slope triangle for PQ. (Note: by definition, the legs of slope triangles are parallel to the axes.)
   b. Draw a slope triangle for QR with third vertex B.
   c. Using the coordinates of P, Q, and R, label the legs of the slope triangles with their lengths. (For example, the leg $PA = x_Q - x$.)

Strategy for proof: To show that P is on the line QR, we will prove that $\angle PQR = 180°$. To do that we will first show that ΔPQA and ΔQRB are similar.

2. Express the lengths of the vertical legs in terms of the x-coordinates and m.

3. Calculate the ratios RB/QB and QA/PA in terms of m and/or the x-coordinates.

4. Prove that P is on the line QR. **Hint:** follow the strategy outlined above!

5. The proof would not have worked if QR had been horizontal or vertical, because we would not have had slope triangles. How do we know QR is neither horizontal nor vertical? **Hint:** remember that we obtained Q and R by using the equation $y = mx + b$, and that we are looking at the case where $m \neq 0$.

A similar proof would work if x is between x_Q and x_R, or if $x > x_R$. (Just switch the coordinates around, and follow exactly the same logic and calculations.) So we have shown that if we start with any two points Q and R on the graph of $y = mx + b$, *all other points P on the graph are on the line QR*. This is Result 1, however we still have a particular case to address:

6. *If $m = 0$*, then $y = b$. Explain why in this case, all the points of the graph are on a straight line. **Hint:** use Fact PE to show that any point with coordinates (x, b) is on the line parallel to the x-axis through the point $(0, b)$.

Result 2: For a given non-vertical line, there is a unique pair of parameters m and b such that all the points on the line satisfy the equation y=mx+b

Again, we will prove this in two cases: first a line that is neither horizontal nor vertical, then a horizontal line. If the line is neither horizontal nor vertical, it has slope triangles.

Strategy for proof: We will show that all slope triangles are similar to each other. Then we will use that fact to write an equation.

1. Draw a line that is neither horizontal nor vertical. Draw an arbitrary slope triangle on it. Label its horizontal leg with its length p, and its vertical leg with its length q.

2. Draw any other slope triangle for this line, with legs respectively r and s. Explain why $s/r = q/p$. **Hint**: you can use Facts T and AA to show that the slope triangles are similar.

In other words, for this particular line, the slope obtained from *any* slope triangle is always the same. Let us call this slope m.

3. Make a new figure, consisting of x- and y-axes and a line that is neither horizontal nor vertical. (This represents the same line as above.) Since the line is not vertical, it will meet the y-axis at a point — call it (0, b).
   a. Label the y-intercept on your figure.
   b. Put a generic point P(x,y) on the line (say with $x > 0$).
   c. Draw a slope triangle for the y-intercept and P.
   d. Label its legs with their lengths in terms of x, y, and b.

4. Write "rise / run = slope" for this particular slope triangle, and solve for y.

If you did this correctly, you have shown that *for a generic point P (x, y) on our line, we have the relationship $y = mx + b$*. This is Result 2, however we need to address a couple of details:

5. The y-intercept is a point which we could not have used for P(x, y) in #3. Explain why that point too satisfies the equation.

6. *If the line is horizontal*, it must meet the y-axis at some point B. Say the coordinates of B are (0,b). What is the equation of the line? **Hint:** use Fact P.

The Proof!

For every pair of parameters m and b, Result 1 tells us that all points whose coordinates satisfy $y = mx + b$ belong to a unique non-vertical line. Result 2 tells us that any point on this line satisfies $y = mx + b$ for a unique pair m, b. Therefore, a point is on the line if and only if its coordinates satisfy the equation.

Likewise, for every non-vertical line, Result 2 tells us that there is a unique pair of parameters m and b such all the points on the line satisfy $y = mx + b$. Result 1 tells us that there are no points off the line that satisfy this equation. Therefore, a point is on the line if and only if its coordinates satisfy the equation. Q.E.D.

Notes and Acknowledgments

Students generally never see a proof of this, because they learn about the equations of lines and the graphs of linear functions before they learn enough geometry for the proof. By the time they know the geometry, we rush on to other topics. Perhaps this is just as well, as the logic of the proof is subtle, and in any case students do not have a burning desire to see the proof.

For many teachers, it is helpful to first go through the arguments presented here with specific numerical examples.

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