

Space

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Computing Transformations

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Teacher Notes

These lessons provide the framework of an approach to computing geometric transformations in a precalculus level course, starting with complex numbers, and ending with matrices.

The lessons assume a fair amount of teacher explanation and class discussion, and should not be expected to teach themselves. The unit is based on material I teach at The Urban School of San Francisco, in my Space class, an advanced high school geometry elective. See www.picciotto.org/math-ed/space.

Prerequisites

Students should have been introduced to complex numbers, and be familiar with their rectangular and polar representations. At The Urban School, that introduction happens towards the end of our Algebra 2 course. See our Complex Numbers unit at www.picciotto.org/math-ed/alg-2/complex.pdf.

Students should also have been introduced to matrices and matrix multiplication. I use the approach in the University of Chicago School Mathematics Project *Advanced Algebra* book.

Technology

Some lessons assume the use of the Texas Instruments TI-89 Titanium calculator, including some of its CAS capabilities. It is of course possible to do much of this work with other graphing calculators or computer software. (If you create a version of this packet using some other technological support, I will gladly credit you and post it alongside this version on picciotto.org.)

Review: Slope of Perpendicular Lines

This lesson reviews the “opposite reciprocal” result, but also proves it. It also includes a reminder of similar triangles. All of this will be needed in the subsequent pages.

Complex Numbers

This lesson combines a (perhaps too sketchy) review of complex numbers with an introduction to two key questions we address in this unit:

- ◇ How do we compute the coordinates of images of points under various transformations?
- ◇ How do we use composition of transformations we know how to compute, in order to figure out more challenging computations?

Note that #3 does not require knowing about multiplication in polar form. (This fact will be important later.)

#4. Multiply by $(1, \theta)$ -- a complex number on the unit circle

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See also the corresponding Cabri figure at <www.picciotto.org/math-ed/space>.

The heart of the proof is SAS similarity between $\triangle ONP$ and $\triangle ORU$. Avoid a circular argument! $U = P \cdot R$ follows from the argument in #2 which is generalized in #7. The right angle at $\angle R$ follows from the argument involving slopes that we used in the "Complex Numbers" lesson.

Computing Any Isometry Using Complex Numbers

The five-step method for #7 is: translate so the y-intercept is at the origin, rotate so the line coincides with the x-axis, reflect, rotate back, translate back. Assuming $\theta = \tan^{-1}(2)$, this calculation will do it:

$$(\text{conj}(((3+2i) - 5i) (\cos(-\theta) + i \sin(-\theta))))(\cos(\theta) + i \sin(\theta)) + 5i$$

Matrices

The remaining lessons on the one hand replicate the approach we used with complex numbers, using matrices, and on the other hand introduce increasingly efficient ways to use the TI-89 calculator to actually do the computations .

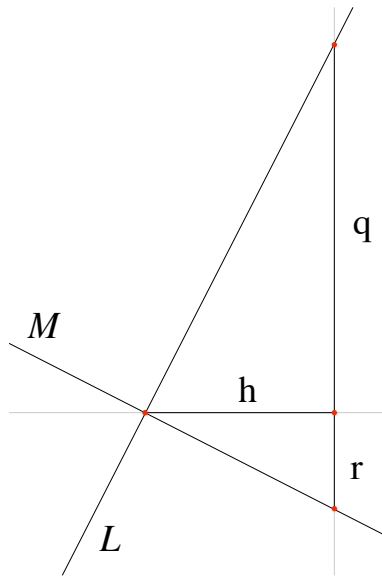
Having students themselves find the matrices for the transformations gives them a sense of control of this technique.

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Review: Slope of Perpendicular Lines

0. What are the three ways to show triangles are similar?



1. Consider two perpendicular lines L and M . What is the relationship between their slopes? Use similar triangles in the figure above to find it out.

2. Now prove the converse of this result. ("If the slopes of two lines are ... then the lines are perpendicular.") You can do it by drawing two congruent right triangles.

Complex Numbers

Reminder:

You can think of a complex number as a vector that starts at the origin.

Rectangular form: $a+bi$, with $i^2 = -1$.

a is the real part. b is the imaginary part. (a,b) is the endpoint of the vector.

Polar form: (r, θ)

r is known as the radius, the modulus, the magnitude, or the absolute value of the number.

θ is the angle or argument.

1. Use basic trig to convert from polar to rectangular form and vice versa. (Draw an example on graph paper to help you remember how to do this. It is easiest to do it in the first quadrant)
2. Complex addition: $(a+bi)+(c+di)$.
 - a. Rearrange the terms so that the real part is first, and the imaginary part last.
 - b. Draw an example of this on graph paper, and explain how this addition works like vector addition.
 - c. How would you use complex numbers to find the image of (a, b) by a translation (v, w) ?
3. Complex multiplication: simple cases, and how they relate to transformations. For each example below, draw an example on graph paper.
 - a. If k is a real number, distribute $k(c+di)$. How does the resulting vector (the result of your calculation) compare with the original $(c+di)$? This transformation is called a *dilation* with factor k and center at the origin.
 - b. Distribute $i(c+di)$. Write the result the usual way, with the real part first, and the imaginary part next. How does the resulting vector compare with the original? What is this transformation called? (Hint: pay attention to the slopes.)
 - c. Distribute $ki(c+di)$. How does the resulting vector compare with the original? This transformation does not have a name. It is the *composition* of the previous two.

In Math 3, you learned that $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$.

4. How would you use complex numbers to find the image of (a, b) by a *rotation* of θ° centered at the origin?

Your teacher will explain how you can use complex numbers to find the image of (a, b) by a rotation of θ° centered at (p, q) .

5. Test this approach on some examples.

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Complex Numbers Practice

A point in the plane can be thought of as a _____ number, just like a point on a number line can be thought of as a _____ number.

1. Write the following three famous points in $a+bi$ (rectangular) form:

a. $(1, 45^\circ)$

b. $(1, 60^\circ)$

c. $(1, 90^\circ)$

In the following exercises, call these points respectively s , t , and i .

2. s , t , and i are all on a certain geometric figure. What am I referring to?

3. Using your calculator, or not, compute the following, and show the results on graph paper:

a. $s+t$

b. $t+i$

c. $s+i$

d. $s \cdot t$

e. $t \cdot i$

f. $s \cdot i$

4. Describe the results of the computations above, using the words *translation* or *rotation*. There are two ways to do this in each case.

5. Explain how to use complex numbers to translate a point (x, y) by a vector (v, w) .

6. Write the coordinates of the point $(1, \theta)$ in $a+bi$ form.

7. Explain how to use complex numbers to rotate a point (x, y) by an angle θ around the origin. (Use $a+bi$ form.)

8. Use the answer to the previous problem to give the rectangular coordinates of the image of (x, y) after a rotation of θ around the origin.

Complex Multiplication

We will use geometry to prove that $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$, starting from the definition of i as the square root of -1 , and the distributive law. (In Math 3, we accepted without proof that all these ideas worked with each other.)

Working an Example

Say that $P=3+2i$ and $Q=2+2i$.

1. Plot P and Q on graph paper. Label the origin O . Draw the vectors OP and OQ .
2. Explain this: $P \cdot Q = (3+2i)(2+2i) = 3(2+2i) + i(2+2i)$
Hint: it may be easier to think about this if you write it $(3+2i)Q = 3Q + 2iQ$
3. On your figure, draw vectors for $3Q$, iQ , $2iQ$, and $3Q+2iQ$. Label their endpoints respectively R , S , T , and U .
4. Explain why $U = P \cdot Q$. (Hint: see #2.)
5. Check that this is right for this example, by calculating $P \cdot Q$, using the distributive law and the fact that $i^2 = -1$.

The Proof

Up to this point, we worked with the rectangular form of P and Q , and focused on a particular example. To prove the result we are after, we will now switch to polar form, and think of P and Q as generic points, with $P=a+bi$ and $Q=c+di$.

6. Say that $P = (r_1, \theta_1)$, $Q = (r_2, \theta_2)$, and $U = (r, \theta)$. Label lengths and angles on the figure accordingly. (If it would be helpful, make a new copy of the figure you created for #1-5.)
7. Check that $U = P \cdot Q$, generalizing what you found in #4.
8. Drop a perpendicular from P to the x -axis and call the point of intersection N .
9. In terms of a , b , c , d , r_1 , and/or r_2 , find the lengths of all segments on the figure.
10. Prove that $\triangle ONP$ is similar to $\triangle ORU$. What is the scaling factor?

You now have all the ingredients you need for the proof.

11. Prove that $r = r_1 \cdot r_2$ and $\theta = \theta_1 + \theta_2$

Computing Any Isometry Using Complex Numbers

Calculator shortcuts:

- ◇ Converting from rectangular to polar: for example $3+4i \rightarrow$ Polar (in the Catalog under P)
- ◇ Converting from polar to rectangular: it's simply $r \cdot \cos(\theta) + r \cdot \sin(\theta) \cdot i$, or for example $(3 \angle 45) \rightarrow$ Rectangular (in the Catalog under R)

1. Make sure you know how to compute the image of a point (a,b) under any translation or reflection, using complex numbers.
 - a. Translation by a vector (v,w)
 - b. Translation by a vector (r,θ)
 - c. Rotation by θ degrees around the origin.
 - d. Rotation by θ degrees around a point (p,q)

What we are missing is a method for computing reflections.

2. The image of (a,b) after reflection in the x-axis is _____.
3. The image of $a+bi$ after reflection in the x-axis is _____.

This image is called the conjugate of $a+bi$, which the TI-89 will return if you enter **conj(a+bi)**.

Three-step method: We will find the image of (a,b) in a line $y=mx$, by rotating so the line lies on the x-axis, then we'll reflect across the x-axis, then we'll rotate back.

4. What is the angle between $y = 2x$ and the x-axis?
5. What is the reflection of the point $(3,2)$ across the line with equation $y=2x$?
6. Now let us reflect $(3,2)$ in the line $y=x$.
 - a. Predict the coordinates of the image
 - b. Check whether the three-step method gives you that answer.
7. Work out a strategy (five steps) to reflect $(3,2)$ across the line with equation $y = 2x+5$. Find the coordinates of the image.
8. Work out a strategy to glide-reflect $(3,2)$ with mirror $y=x+4$ and vector $(5,5)$

You can now compute the results of any isometry! However this method has shortcomings:

- ◇ It is tricky to keep track of parentheses and order of operations.
- ◇ It is inconvenient when calculating the image of a polygon, as the calculation has to be done for each vertex.
- ◇ It does not generalize to three dimensions.

All of those concerns are addressed by using matrices. In fact, all computer animation is done by matrix calculations of geometric transformations. This is what we will learn next.

Computing Some Transformations with Matrices

We will represent points using a vertical (2 by 1) format: $\begin{bmatrix} x \\ y \end{bmatrix}$. An n-gon will be represented by

a 2 by n matrix. For example: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ represents a right triangle.

1. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a dilation centered at the origin, with factor k.
2. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the y-axis.
3. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the $y=x$ line.
4. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a 180° rotation around the origin.

In a previous lesson, you found the formula for the image of a point (x,y) under a rotation of θ° around the origin.

5. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a rotation of θ° around the origin.

In a previous lesson, you found a way to use the complex conjugate as part of a three-step process to reflect a point across any line through the origin.

6. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the $y=2x$ line. Note that the matrices are written from right to left in the multiplication.

Matrices on the Calculator

It is tedious to enter matrices over and over. Here is a shortcut, using STO.

$[[\cos(t),-\sin(t)][\sin(t),\cos(t)]] \rightarrow \mathbf{ro}$

$[[1,0][0,-1]] \rightarrow \mathbf{rx}$

You will not need to enter these matrices again.

1. What do these matrices do? You should recognize them without looking at your notes.
2. Explain the calculator's response if you enter
 $\mathbf{ro} \mid \mathbf{t=45}$
 $\mathbf{ro} \mid \mathbf{t=60}$
3. How would you use matrix multiplication to reflect a point (x_0, y_0) around a line that makes an angle θ with the positive x-axis?
 - a. Say this in words, not symbols
 - b. Write the matrix multiplication, remembering that you go from right to left.
4. Test your answer with the matrices we entered, using 45° since we know what the result of that reflection would be. Here is how you would enter this:
 $(\mathbf{ro} \mid \mathbf{t=45}) * \mathbf{rx} * (\mathbf{ro} \mid \mathbf{t=-45}) [[x_0][y_0]]$
 The calculator should respond with the coordinates (y_0, x_0)
5. If a line has equation $y = 3x$, what angle does it make with the positive x-axis?
6. What is the image of $(4, 5)$ across the line $y = 3x$?

Matrices and Translation

The shortcoming of using matrix multiplication in the way we have been is that it does not work for translations. This problem is solved if we represent points as 3 by 1 matrices, like this:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

1. Find a 3 by 3 matrix M such that $M \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a translation by vector (v,w) .

2. Find 3 by 3 matrices for

- Reflection in the x -axis
- Rotation around the origin by an angle θ .

3. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a θ° rotation around a point (p,q) . Remember that the matrices are written from right to left in the multiplication.

4. Find a sequence of five matrices J, K, L, M, N such that $N \cdot M \cdot L \cdot K \cdot J \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a reflection across the line $y=mx+b$.

Better Calculator Shortcuts

Setup

- To create an easy-to-use *function* that will output a 3 by 3 translation matrix, use **Define**, which is in the F4 menu:
Define $\text{tr}(v,w)=[[1,0,v][0,1,w][0,0,1]]$
 From now on, $\text{tr}(3,4)$, for example, will output the translation matrix for vector (3,4).
- Use the same method to make functions for:
 - a 3 by 3 matrix for a rotation of t° around the origin -- $\text{ro}(t)$. (You may need to delete the matrix ro first.)
 - a 3 by 1 matrix for a point with coordinates (a,b) -- $\text{pt}(a,b)$
- For reflection across the x-axis, you don't need a function, as it is always the same matrix. If you don't yet have it, use **STO** to store the 3 by 3 matrix under the name rx .
- For convenience, you should also store a 3 by 1 matrix for (x,y) under the name xy .

Practice

- To rotate (2,3) 53° around (4,3), do this:
 $\text{tr}(4,3)*\text{ro}(53)*\text{tr}(-4,-3)*\text{pt}(2,3)$
- To find a matrix for rotation of any point (x,y) t° around (a,b):
 $\text{tr}(a,b)*\text{ro}(t)*\text{tr}(-a,-b)*\text{xy}$
- Reflect (2,3) across the line $y=x+4$
 - "manually" on graph paper
 - using matrices on the calculator (you need five matrices in reverse order!)
 Hopefully you get the same answer both ways.

Polygons

- Store a 3-rows, n-columns matrix ($n \geq 3$) for a polygon, as po . I recommend:
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

To draw your polygon, download the dr program from the conference. (You'll need a USB cable to connect the calculator to the computer. Use the program TI Connect.)

- You can draw your polygon by entering: $\text{dr}(\text{po})$
- You can draw images of your polygon by entering expressions like $\text{dr}(\text{ro}(45)*\text{po})$