Quadratic Functions Teacher Notes

This unit is adapted from *Algebra: Themes, Tools, Concepts* by Anita Wah and Henri Picciotto, Lessons 7.2 and 13.1-13.3. The book is available for free download at
www.MathEducationPage.org/attc/
Also available on the Web site: a Teachers' Edition with answers and additional notes, as well as some quizzes and tests.

Many thanks to Vinci Daro and Ann Shannon for their suggestions.

Introduction

Student access to graphing technology is very helpful for this unit, but it is not absolutely required. If students do not have such access, teacher projections of the relevant quadratic graphs would be helpful to class discussions.

Quadratic functions are introduced as a model for two different "real-world" situations. In both situations, the quadratic functions appear alongside linear functions. This serves as a review of linear functions, and it helps ground the quadratic models in an accessible context. (As a general rule, it is easier for students to make sense of new ideas when they are not divorced from ideas they are already familiar with.)

This approach to quadratic functions is consistent with the Common Core major shift in the teaching of algebra: more emphasis on modeling, less emphasis on symbol manipulation; more emphasis on understanding, less emphasis on memorization.

The unit ends with a general introduction to the intercepts and vertex of quadratic functions when the equation is in factored form. This is the most accessible form, and it provides a visual foundation for later work with standard or vertex form.

Lesson 1, Part 1: Square Windows

You may start the lesson with students reading the introduction, and doing #1. If there are discrepancies among their answers to #1, stop them and have a whole-class where students explain the three types of windowpanes to each other in order to justify their answers to #1.

It is not necessary to teach any math up front; in fact it would be detrimental and intimidating. Students can start this lesson if they can count. There will be plenty of time to discuss the math later on, when they are familiar with the context under study.

#1: This is just to confirm that students understand the three types of panes. The rest of the lesson concerns square windows.

#2: Make sure students understand the question by having students explain the given example to their neighbors or to the class. There are many solutions, and it is not necessary to find them all. Still, you should encourage creativity and praise new solutions. The more solutions are found, the easier it will be to fill out the table in #3.

#3-4: Students should be allowed to use numbers found by other students when working on #2 in order to fill out the table. Allow for a full discussion of the patterns of change. You may tell the students that the data will be graphed in the next lesson, and ask if they can predict what the three graphs may look like.

Lesson 1, Part 2: Graphing Square Windows

Start by reminding students about the table they made in Part 1.

#1a: After students have had a chance to work on this question for a few minutes, have them present their answer to a neighbor, and then have a whole-class discussion of this. Understanding this in words is the strongest foundation for being able to answer the questions in 1b.

#1b: Filling out this table of large numbers, after the discussion of 1a leads to formulas for n by n windows. Refrain from explaining this to the class prematurely. See how far things can get based on the work they have done so far. If it is necessary for you to explain how to get the formulas, it will be a lot easier for students to hear and understand your explanation if they have struggled with these questions first.

#2: Have students compare their answers to this question. If they have disagreements, those can be resolved first by finding the sums numerically in the tables from both parts of this lesson, then by discussing what result should be expected, and finally by checking the algebraic manipulations. Doing all three is far better than choosing one approach to checking, because each approach will tend to reach different types of students. And for stronger students, it is important to see the connections between three ways of thinking about this. (The three ways: numeric, geometric, algebraic.)

#3: The points from the data table can be plotted in different colors, or if that is not possible, using different symbols for each graph (e.g. •, +, x). To make the graphs more legible, it is helpful to connect the points in each one, even though the graphs are about whole numbers.

#4: Students should note that the third graph is not linear. This is the first parabola in the unit.

#5: Do not expect very sophisticated answers at this early point in the unit. The main idea is that for the inside panes, the numbers increase faster and faster, and thus the graph cannot be a straight line.

#6: This is an optional extension puzzle. Students can try to outdo each other’s solutions.

Lesson 2: Rectangular Pens (Width as a Function of Length)

This lesson (and the next two) only makes sense if length can be shorter than, equal to, or longer than width. Make this clear up front, as students may have been told in the past that length is always greater than width.

Start by a discussion of the introduction. For example, you may say something like: “You have 28 feet of fencing, but 10+4=14, and 10·4=40. What does this figure have to do with the problem?” Have students discuss this. If necessary, you may hint by asking what the perimeter of the rectangle in the figure is.

#1: Setting this up with length and width axes allows students to see an interesting graphical relationship between those two variables. Don't tell the students how to find the required dimensions. Instead, take examples found by some students and discuss whether they have a perimeter of 28. If yes, that is a correct solution. If not, they need to try some other numbers.

#2: Students may find it difficult to come up with an equation. A good hint is to remind them of how they found the possible pens. Understanding this is the key to seeing the algebraic relationship.

#3-5: These questions require students to go back and forth between what they see on the graph, and what the meaning of those features of the graph is in the "real-world" context the graph models.

#6: This establishes that all the areas are not equal, and sets the stage for the next lesson.

Lesson 3: Rectangular Pens (Area as a Function of Length)

#1: It is not necessary to settle this question right away. Its purpose is to make clear what the lesson is about. In fact, it can probably be answered simply by looking at the graph the students annotated in Lesson 2, and this lesson will confirm that answer.

#2: Students should have a lot of the information they need to make this graph from the work they did in Lesson 2. In fact, looking at those numbers should help students set appropriate scales on their graphs. You may precede this with a discussion of the scales, so that students do not waste time trying to make their graphs with axes that have inappropriate scales.

#3: Again, this is about going back and forth between the graph and its interpretation.

#4-6: This is a process of getting to a more and more abstract and symbolic generalization of the model explored in Lessons 2 and 3. There is no shortcut to this: a teacher explanation, even an excellent one, will only reach very few students if it is a substitute for reflection and discussion. After some time working on these questions on their own, ask students to discuss these problems with their neighbors, and then lead a whole-class discussion. A teacher explanation at the end of that discussion will almost certainly reach more students than an attempt to teach them how to do these problems proactively.

Lesson 4, Part 1: Against the Garage
Lesson 4, Part 2: Garage Graphs

This lesson parallels the previous two lessons, with a somewhat different setup.

After students have read the introduction and answered #1-2 of Part 1, make sure the sketches they made are correct before having them go on with #3-5 and then with Part 2.

Note that students will have to decide on the scale for the graphs. If necessary, you can support them by giving some class time to a discussion of that.

Lesson 5: The Zero Product Property

You might start the lesson by telling students you are thinking of two numbers, a and b, such that $ab=24$. Tell them that the first students who guess just one of your numbers will get a handshake. (Or something.) Whatever guesses they make, tell them they are wrong.

Then say you are now thinking of two numbers, *a* and *b*, such that *ab*=0. Shake the hand and congratulate the first student to say that one of your numbers must be 0. Express amazement at the fact they are psychic and were able to read your mind. Ask the class how they think this was even possible!

After students have worked on #1-4, you may have a discussion during which students will share their strategies for solving #4, and then let them work on #5-7.

Lesson 6: Symmetry of Parabolas

The graph is the same one students made in **Lesson 4, Part 2**.

However the formula shown may be different from the one used by your students, as most of them probably ended up with $y = x(28–2x)$. If the question comes up, explain how those two equations are equivalent by using the distributive property. Do not spend too much time on this, as it is not the subject of the lesson. If you're concerned about your students' weakness with distributing and factoring, save that conversation for a future session that focuses on that, perhaps with the help of manipulatives and an area model.

Again, this is not the main point of the lesson, but you could take a little time to ask students to review the meaning of the intercepts and vertex in this graph, as discussed in **Lesson 4, Part 2**.

The main point of the lesson is to learn how to graph an equation in the form $y=a(x-p)(x-q)$. If students have access to graphing technology, they should only use it to check their answers in this lesson.

#2-5 will probably be very challenging, but resist the temptation to explain everything up front. See how much students can achieve by themselves and with each other's help. If necessary, you can provide hints. Before going on to #6 would be a good time to collect student answers to #2-5, along with explanations of how they were arrived at.

#6-7 will provide a good test of how well students can apply the strategies shared after #2-5.

#8-9 are challenging applications of the ideas learned in #2-7.

Lesson 1, Part 1: Square Windows

You will need: graph paper

Three Types of Panes

The A.B. Glare window store has started selling a new kind of window. These windows can be made to order by combining three types of square windowpanes. Each pane measures one foot on each side. The three types of panes are shown in the figure below: corner panes, edge panes, and inside panes.

A 3 foot by 3 foot window is shown in the figure below. It was made by putting together 4 corner panes, 4 edge panes, and 1 inside pane.

1. Sketch a 4 foot by 5 foot rectangular window. How many panes of each type were used to make it?

Use this grid paper to work on problems #2 and 3.

2. **Exploration.** A builder is going to build a cafeteria that will have only square windows. The windows will be made of the panes described above. The builder decides to consider various *combinations of square windows* that will give a total area of exactly 72. For example, two 6 by 6 windows would work, because $6 \cdot 6 = 36$, and $36 + 36 = 72$.
   a. Find a different combination of square windows that will give an area of exactly 72 square feet.

   b. For each window in that combination, find the number of each type of pane the builder will need.

3. To save time when customers ask for square windows, Lara is assembling kits with the correct number of corner panes, edge panes, and inside panes to make square windows of various sizes. Complete this table to show how many panes of each type are needed for a 2 ft. by 2 ft. window, a 4 by 4 window, and so on up to a 10 by 10 window.

Square window	Corner Panes	Edge Panes	Inside Panes
2 by 2			
3 by 3	4	4	1
4 by 4			
5 by 5			
6 by 6			
7 by 7			
8 by 8			
9 by 9			
10 by 10			

4. Study the table. Describe some of your observations about the pattern of change for each type of pane.

Lesson 1, Part 2: Graphing Square Windows

In Part 1, you made a table that showed how many of each type of windowpanes you would need for square windows of different sizes. In this lesson, you will find patterns, generalize, and make predictions. You will also look at that same information with the help of graphs.

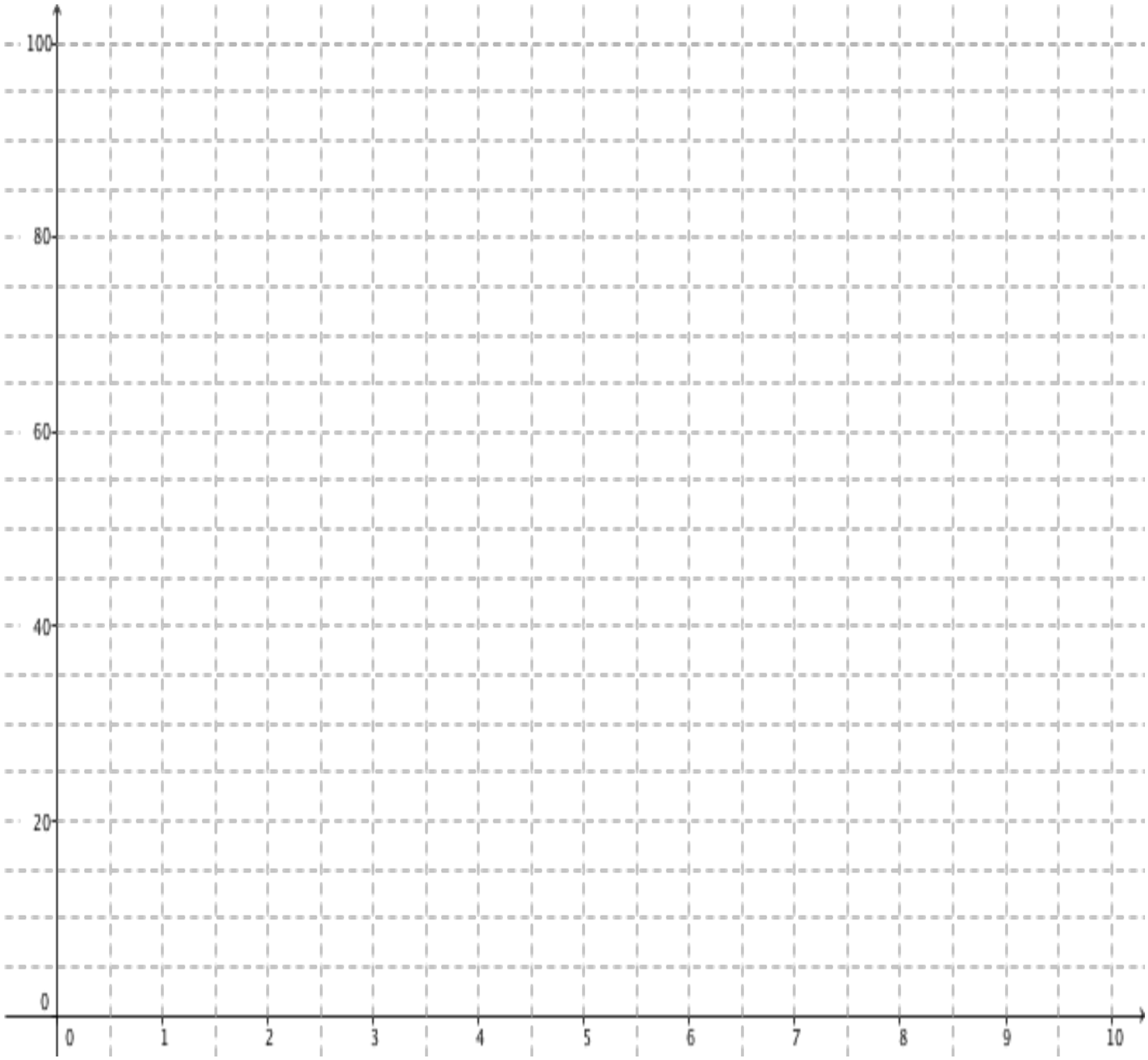
1. **Generalization.**
   a. Explain in words how to find the number of panes of each type in an n by n window. Explain each with reference to a sketch of such a window.

   b. Complete the following table by using your generalization in 1a to predict the number of planes of each type that are needed for these larger windows.

Square window	Corner Plane	Edge Pane	Inside Pane
20 by 20			
30 by 30			
50 by 50			
100 by 100			
n by n			

2. Add up the algebraic expressions for the numbers of each type of pane. If you did your work correctly, the sum should be a simple one. What does your result represent in the context of the windows?

3. On the same set of axes, below:
   a. Graph the number of corner panes as a function of the length of the side of the window. For example, since a 3 by 3 window uses 4 corner panes, the point (3,4) would be on your graph.
   b. Graph the number of edge panes as a function of the side length.
   c. Graph the number of inside panes as a function of the side length.



4. Study your graphs. Which are linear? Which grows the fastest? Explain.

5. **Summary:** Compare the algebraic expressions you wrote in problem 1 with the graphs you sketched in problem 1. For each type of windowpane, explain how the expression and the graph both represent the same pattern. Pay attention to how the numbers change as n increases.

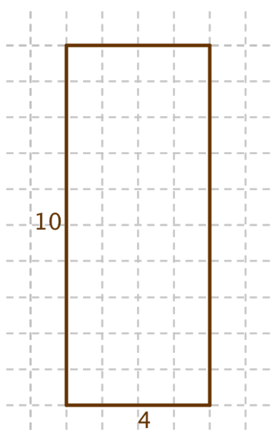
6. **Challenge:** Assume that you try to make any number of square windows, with the goal of having *as few panes as possible left over*.
   a. If you start with 100 panes of each type, what size windows should you make? What will be left over?

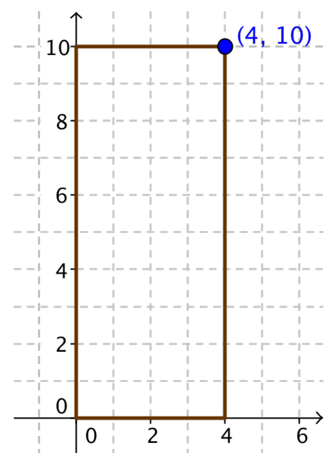
   b. Compare your answers with other students.

Lesson 2: Rectangular Pens (Width as a Function of Length)

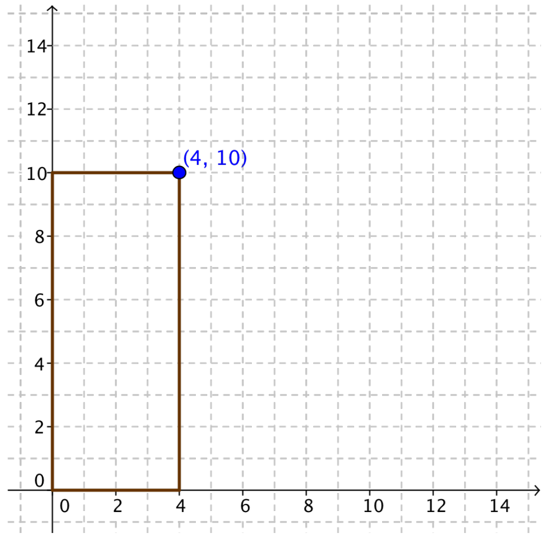
You want to make a rectangular pen for Stripe, your pet zebra. You have 28 feet of fencing. There are many possible dimensions for this pen. One possible pen, 10 feet wide by 4 feet long, is shown in the two figures below, first on grid paper, then in a graph. The coordinates of the point on the top right give us the length and width of the pen.

In this lesson, you will investigate how the length and width change in relation to one another if you keep the perimeter constant.





1. a. Using the axes below, draw at least five more pens having a perimeter of 28.
Make sure that the whole pen is in the first quadrant, and one vertex is at the origin, as in the given example.



b. The upper right hand corner of the pen on the figure has been marked with a • and labeled with its coordinates. Do this for the pens you drew. Then connect all the points marked with •. Describe the resulting graph.

2. a. Make a table showing all the coordinates you marked with a • on your graph.

b. Look for a pattern and make three more entries in the table.

c. Write an equation for the function described by your graph and table.

3. The point whose coordinates are (4,10) is on the graph.
   a. What does the sum of these numbers represent in this problem?

   b. What does the product represent?

4. a. What is the greatest possible length of a pen? How can you see this on your graph?

   b. How many rectangles are possible if the dimensions are whole numbers? How many are possible otherwise?

   c. Explain why the graph should not be extended into quadrants II and IV.

5. If you increase the length by one foot, does the width increase or decrease? Does it change by the same amount each time? Explain.

6. Write the area of the corresponding rectangle next to each of the points marked with a • on the graph from problem 1.

Lesson 3: Rectangular Pens (Area as a Function of Length)

You want to make a rectangular pen for Stripe, your pet zebra. Even though Stripe takes a lot of walks around town, you want to make sure she has as much space as possible inside the pen.

1. You have 28 feet of fencing available. If you use all of it to make the pen, what is the biggest area possible?

In the previous lesson, you may have noticed that the area of the rectangles changed even though the perimeter remained constant. In this lesson, you will investigate how the area changes as a function of length, if you keep the perimeter constant.

2. Make a graph of the area of the rectangle as a function of the length of the rectangle. Show length on the x-axis and area on the y-axis. Connect the points on your graph with a smooth curve.

This curve is called a *parabola*. Its highest point is called the *vertex*.

3. a. Label the highest point on your graph with its coordinates.
   b. Interpret these two coordinates in terms of the context of this problem.

   c. Where does the graph cross the x-axis? What do the coordinates of this point mean in the context of this problem?

   d. If you increase the length by one foot, does the area increase or decrease?

   e. Does it change by the same amount each time? Explain.

4. **Summary**.
   a. Describe in words how you would find the area of the rectangular pen having perimeter 28 if you knew its length.

   b. If the perimeter of a rectangular pen is 28 and its length is L, write an algebraic expression for its area in terms of L.

   c. If you had 28 feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

5. **Generalization**. Say the perimeter of a rectangle is P and its length is L. Write the following expressions in terms of P and L. (A sketch may help.)

    a. Express the width of the rectangle in terms of only P and L.

    b. Express the area in terms of only P and L.

6. Explain how to find the length that gives the maximum area. Write an algebraic expression for it in terms of P only.

Lesson 4, Part 1: Against the Garage

Assume that you have 28 feet of fencing to build a rectangular pen. You think that you may be able to get a bigger pen if you use the garage wall as one side of the pen. This way you would need to use your fencing for only three of the four sides. In this lesson, you will figure out how much better this is.

1. Make a rough sketch of what this pen might look like.

Call the side of the pen parallel to the wall the *length*, and the distance between the wall and the side opposite the wall x.

2. On your sketch, label the sides x and *length*.

3. Is it possible to get a square pen? If so:
   a. What are its dimensions?

   b. What is its area?

   c. How does it compare with the maximum area if you did not use the garage wall? (see Lesson 3)

4. Work with your neighbors to fill out this table.

x the distance between the wall and the side opposite the wall	**Length** side of the pen parallel to the wall	**Area of the pen**

5. **Generalizations**. Look for patterns in your table, and study your sketch. Express algebraically as functions of x,
   a. the length

   b. the area

Lesson 4, Part 2: Garage Graphs

In Part 1, you made a table about the length, width, and area of a rectangular pen against the garage, using 28 feet of fencing.

1. a. Graph the length as a function of *x*.

b. What are the *x*- and *y*-intercepts?

c. What do they mean in terms of this problem?

2. a. Graph the area as a function of x.

b. What are the x-intercepts?

c. What do they mean in terms of this problem?

d. What are the coordinates of the vertex? (**Hint**: its x-coordinate is halfway between the x-intercepts.)

3. a. What is the maximum area possible?

   b. How does it compare with the area of the square pen? (see #3 in Part 1)

   c. How does it compare with the maximum area if you did not use the garage wall? (see Lesson 3)

4. **Generalization / Challenge:** Imagine you are giving advice to a friend about how to make as big a rectangular pen as possible if they can use a garage wall for one side of the pen. Should they use the garage wall? What value should they choose for x if they have P feet of fencing?

Lesson 5: The Zero Product Property

1. If $ab = 0$, which of the following is impossible? Explain.
   a. $a \neq 0$ and $b \neq 0$

   b. $a \neq 0$ and $b = 0$

   c. $a = 0$ and $b \neq 0$

   d. $a = 0$ and $b = 0$

Zero Product Property:

When the product of two quantities is zero, one or the other quantity must be zero.

2. Explain the Zero Product Property to your neighbors.

An equation like $(x + 6)(2x - 1) = 0$ can be solved with the zero product property. Since the product in the equation is zero, you can write these two equations.

$$x + 6 = 0 \qquad \text{or} \qquad 2x - 1 = 0$$

3. Solve these equations. Write the solutions.

4. There are two solutions to the equation $(x + 6)(2x - 1) = 0$. What are they?

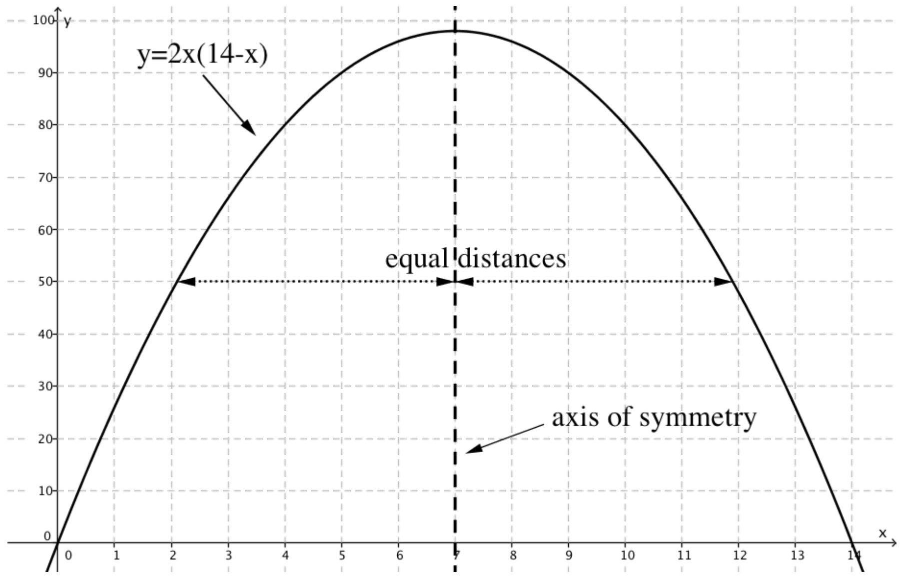
Solve these equations.

5. $(3x + 1)x = 0$

6. $(2x + 3)(5 - x) = 0$

7. $(2x - 2)(3x - 1) = 0$

Lesson 6: Symmetry of Parabolas



Definition: The vertical line through the vertex of a parabola is called its *axis of symmetry*.

The above is the graph for Lesson 4 (area as a function of length, if the pen uses the garage wall.)

1. How far is each *x*-intercept from the axis of symmetry in the above graph?

The *x*-intercepts are *equidistant* from the axis of symmetry. (They are at an equal distance from it.) As you can see in the figure, this is also true of any pair of points of the parabola that lie on the same horizontal line as each other.

In an equation like $y = 2(x + 3)(x - 4)$, you can quickly find the x-intercepts and the vertex.

2. What is the value of *x* at the *y*-intercept? Substitute this value for *x* in the equation, and find the *y*-intercept.

3. What is the value of y at the x-intercepts? Substitute this value for y in the equation, and find the x-intercepts with the help of the zero product property.

4. If you know the x-intercepts, how can you find the x-coordinate of the vertex? Find it.

5. If you know the x-coordinate of the vertex, how can you find its y-coordinate? Find it.

6. Find the intercepts and vertex for:
   a. $y = 0.5\,(x - 0.4)(x - 1)$

   b. $y = 2(x + 3)(x + 4)$

7. **Generalization:** Explain how you would find the intercepts and vertex for a function of the form

$$y = a\,(x - p)(x - q)$$

8. **Challenge:** Find the equation and the vertex for a parabola with intercepts:
   a. (3, 0), (6, 0), (0, 36)

   b. (3, 0), (6, 0), (0, 9)

   c. (-3, 0), (-6, 0), (0, -9)

   d. (-3, 0), (6, 0), (0, 6)

9. **Challenge:** The vertex and one of the two x-intercepts of parabolas are given. Find the equation and the y-intercept.
   a. vertex: (2, -2). x-intercept: (1, 0)

   b. vertex: (1, -12). x-intercept: (-1, 0)

   c. vertex: (3, 4.5). x-intercept: (6, 0)