Chapter 7 Solving Equations

This chapter focuses on simple equation solving.

New Words and Concepts

By now, your students have done a lot of work to lay the groundwork for the **solving of linear equations**. Familiarity with the manipulation of variables, the ability to combine like terms, to simplify expressions, to handle parentheses and minus signs, are all necessary for effective work with equations. Some students may even have developed equation-solving techniques on their own in the context of working on the "Which is Greater?" Explorations. (They get one more chance to do this at the beginning of this chapter.)

The overview of the different types of numbers (**natural**, **integer**, **rational**, **irrational**, **real**, and **complex**) provides a context for equation-solving.

Teaching Tips

Very few equation-solving techniques are given to the student. The "adding zero" trick works, but it is likely that your students will come up with more efficient methods, such as the following:

- Moving quantities from upstairs in the minus area to downstairs outside, and vice versa.
- Moving quantities diagonally (from the minus area on one side to outside the minus area on the other side, or vice versa). This is a shortcut that corresponds to adding or subtracting the same quantity on both sides.
- Moving out all the quantities from both of the minus areas, and moving all quantities from the two outside areas into the minus areas. This corresponds to multiplying both sides by –1.

If some of your students have not discovered these techniques, you can demonstrate them on the overhead projector. Whenever possible, credit a student with the discovery of the tricks, or better yet, ask a student to demonstrate.

After this chapter, you may want to skip directly to the lessons on simultaneous equations in Chapter 9, which follow naturally from the work with linear equations. Doing them later, however, has the advantage of offering a review of this material.

Lesson Notes

- **Lesson 1,** Linear Equations, page 84: Do not give away any techniques yet, except for the ones used in the examples. Encourage students to get help from each other if they are having trouble.
- **Lesson 2,** Solving Equations, page 88: The new element here is the presence of parentheses and the need to use the distributive rule in solving. Students may be tempted to do these problems without the blocks. Insist that they solve at least some of them with the blocks, and compare their work with and without the blocks.
- **Lesson 3,** Solving Techniques, page 89: This is the first explicit mention of equation-solving methods.
- **Lesson 4,** Solving Tricks, page 92: This lesson provides an opportunity for the students who are most proficient with the blocks to share their techniques with the whole class, perhaps on the overhead projector.
- **Lesson 5,** Equations and Numbers, page 94: You can point out to students that their own growth as a math student since kindergarten has been taking them to ever increasing number realms.

Exploration 1 Which is Greater?

For each of these problems, try to decide which of the two expressions is greater. If it is impossible to tell which side is greater, try to tell *which values of x make the two sides equal. Which values make the left side greater?*

Some of the problems may be challenging. If you cannot answer the questions, make a note of the problem, and come back to it later.

1. x **?** $2x + 3$
2. $4x$ **?** $4x + 5$
3. $6x$ **?** $7x^2 + 6x - 7$
4. $8x + 9$ **?** $-x^2 + 8x + 9$
5. $3x^2 + 2x$ **?** $3x^2 + 2x - 5$
6. $3x^2 + 4x$ **?** $3x^2 + 6x + 10$
7. $3x + 4$ **?** $5(x + 6)$

Lesson 1

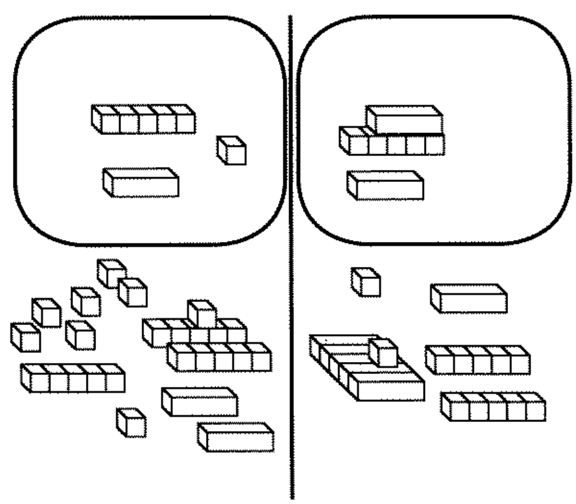
Linear Equations

For each of these statements, indicate always, sometimes, or never true by writing A, S, or N. Explain each answer.

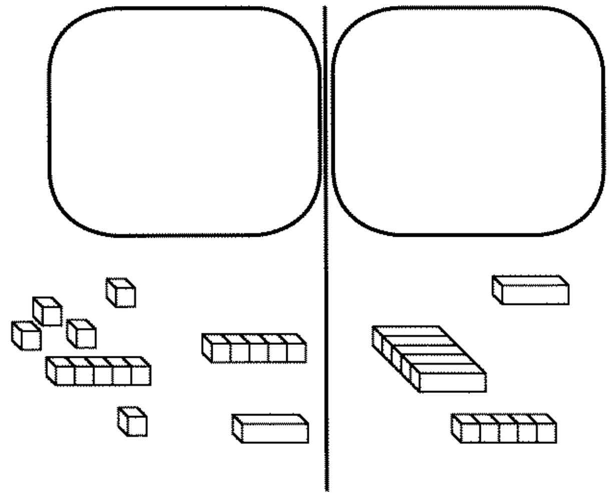
1. $3x = 3x + 5$
2. $3x = 2x + 5$
3. $3x = 2x + x$
4. Write the value of x that makes the statement in problem 2 true. If you did this correctly, you found the **solution** of the equation (you *solved* the equation). Does this equation have other solutions?

Much of algebra is about solving equations. The easiest equations to solve are the ones where there is only one variable, and it is never raised to a power greater than one. These are called **linear equations in one variable**.

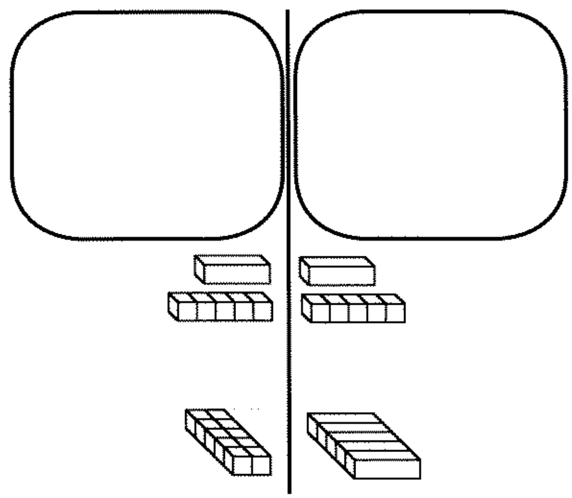
5. Put out blocks to match this figure. Write the equation.
   - To solve the equation, start by simplifying each side, by cancelling opposites.



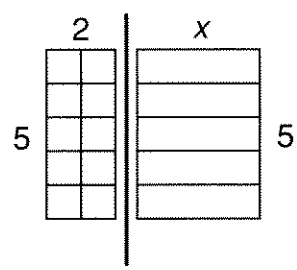
- If you did it correctly, you should have blocks to match this figure.



- Rearrange the blocks like this, matching blocks on the left and right side. Mark the matching blocks in the figure.



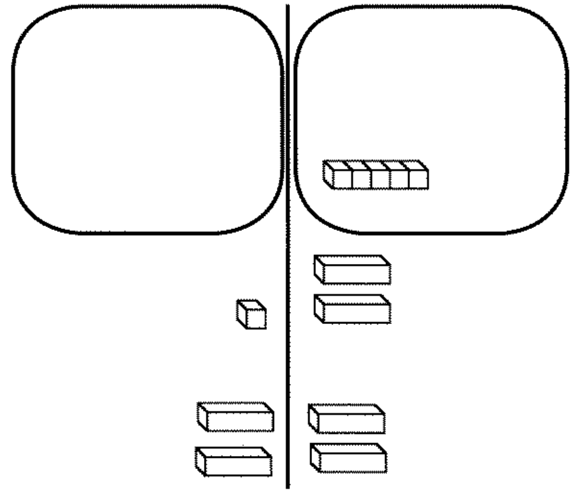
- Look at the other blocks and remember that the two sides are equal. (This is true even though they don't *look* equal. Remember, x can have any value.) You should see the solution to the equation.



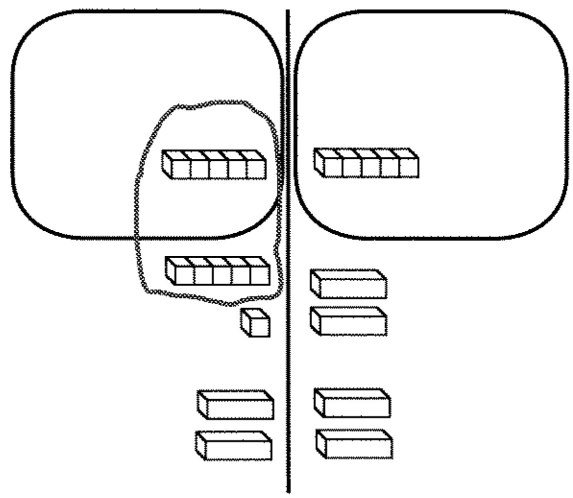
6. Write the solution.

Here is another example. Use the Lab Gear to solve the equation
$2x + 1 = 4x - 5$.

Even with each side fully simplified, and the blocks nicely arranged, it is not easy to tell what the solution is.

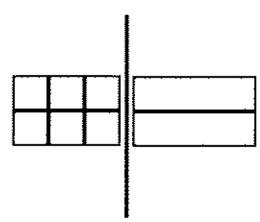


The adding zero trick helps once again.



7. Mark all the matching blocks.

Rearrange the blocks that are left like this.



8. Find the value of x that makes the equation true. Write this solution.

Write and solve these equations.

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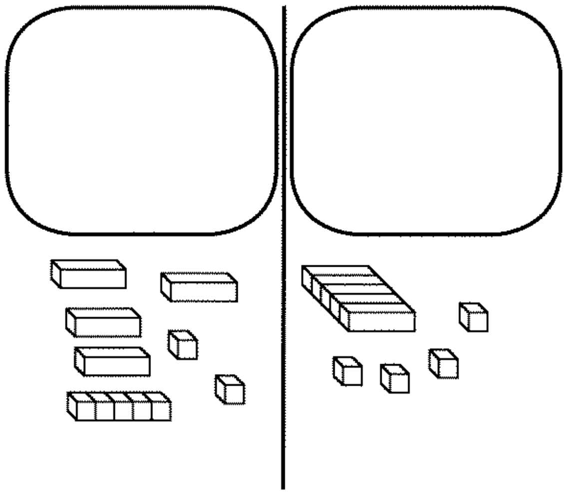

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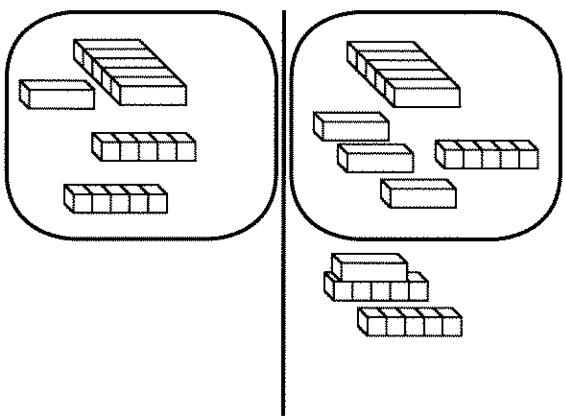

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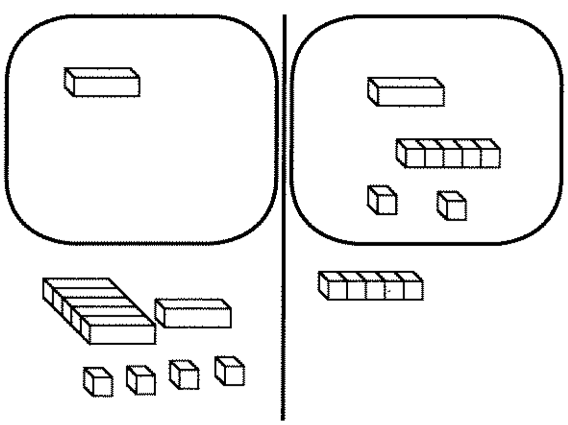

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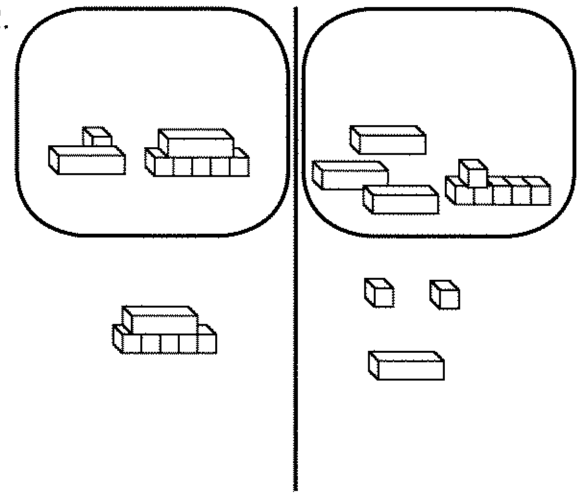

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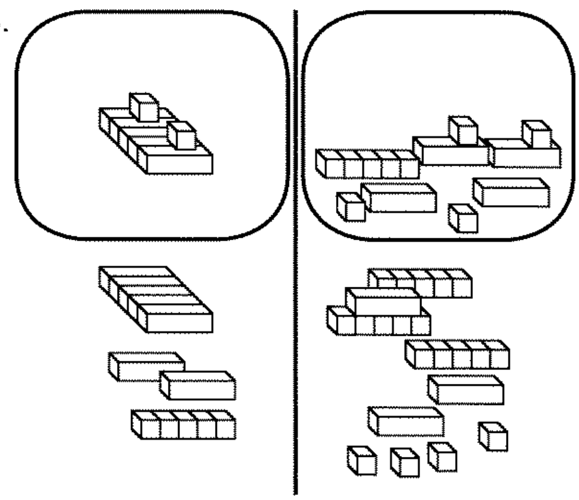


Exploration 2 How Many Solutions?

Think about the number of solutions for each of these equations. Some have one solution, some have two, some have no solutions with rational numbers, and some are identities. (Remember that identities are true for *all* values of the variable.)

Find the solutions. You may use the Lab Gear, but for some of the problems it may not help.

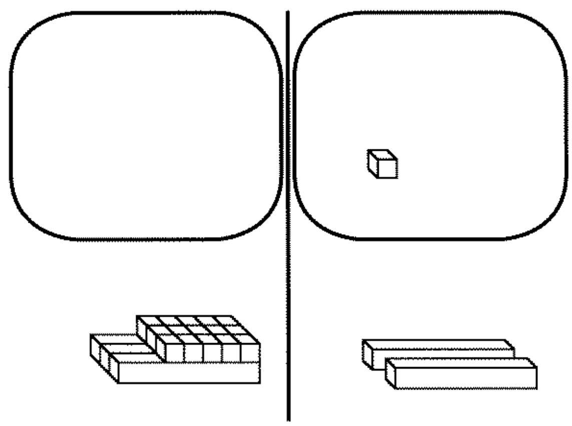
1. $9x + 7 = -2$
2. $2x = 15$
3. $x^2 = 16$
4. $2x + 4x = 6x$
5. $2x + 4x = 8x$
6. $2x \cdot 3x = 6x^2$
7. $2x \cdot 3x = 5x^2$
8. $2x \cdot 3x = 6x$
9. $x^2 = -16$
10. $x^3 = -8$
11. $(x + 2)(x - 3) = 0$
12. $x^2 = 10$

If you had trouble with the last problem, or the last few, discuss them with your classmates. If you still do not understand them, don't worry. You'll come back to them in future lessons.

Lesson 2

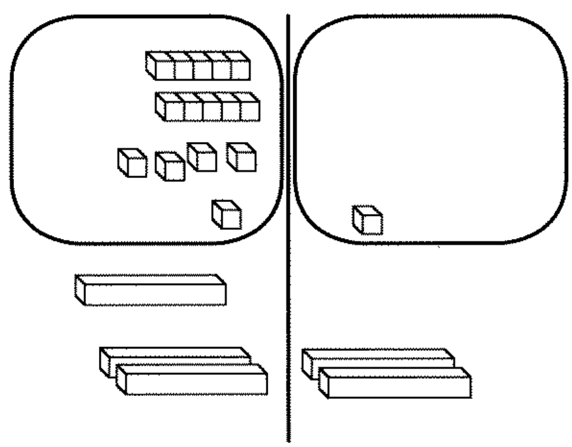
Solving Equations

To solve the equation $3(y - 5) = 2y - 1$ with the Lab Gear, we must have some way to show the multiplication on the left side. This figure shows how.



But to solve the equation, it would be easier to move the upstairs 5-blocks into the minus area, and then reorganize the blocks so matching blocks are near each other.

1. Write the value of *y* that makes both sides equal. (Hint: Start by marking matching blocks.)



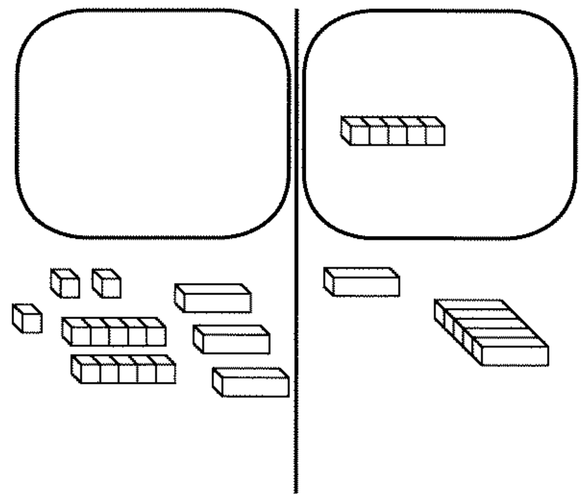
Use the Lab Gear to solve these equations.

2. $3(x + 2) = 15$
3. $2(2x + 5) = 5(x + 1)$
4. $4(2x + 1) = 5(x + 2)$
5. $4(2x - 1) = 5(x + 4)$
6. $3(5 - x) = 9(x - 1)$

Solving Techniques

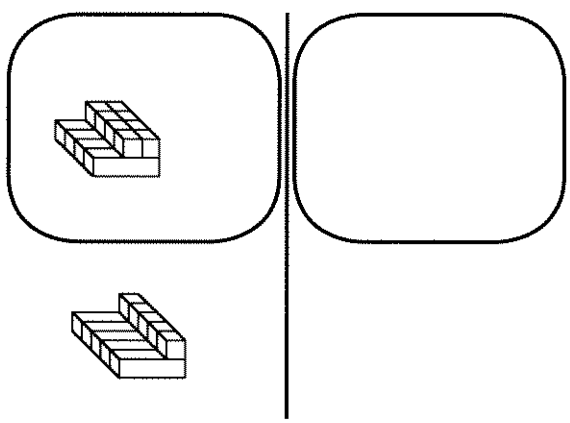
One key to solving linear equations is a technique based on this fact: If two quantities are equal, and you *add the same quantity to both or subtract the same quantity from both*, you end up with equal quantities. This provides you with a method for simplifying equations.

1. Write the equation shown by this figure.
   - Remove three x-blocks from each side.
   - Add 5 to each side and cancel.
   - Finally, form rectangles on both sides.
2. Write the solution to the equation.



For each figure, write the equation, then solve for x. Use the method shown above.

3.



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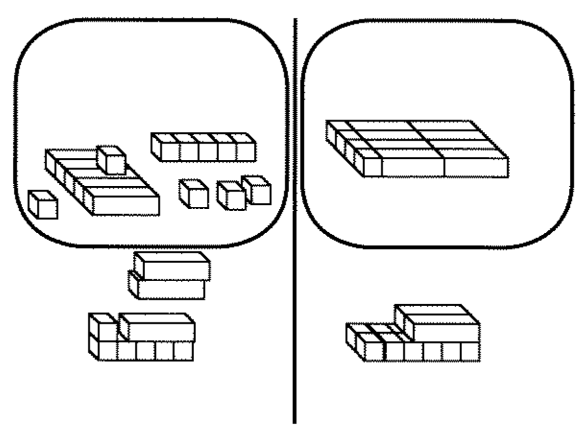
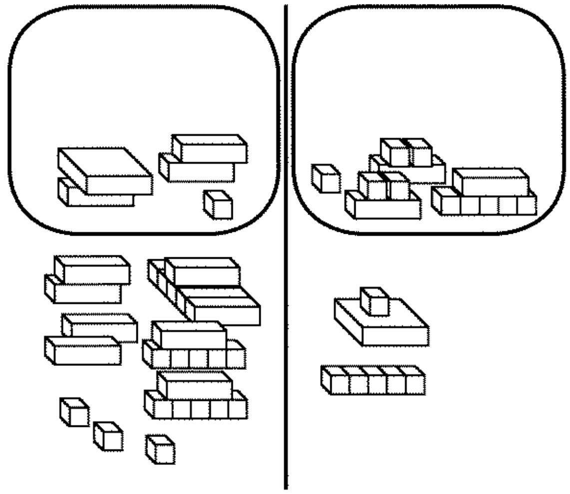


☑ Self-check

3. $x = -5$

5.



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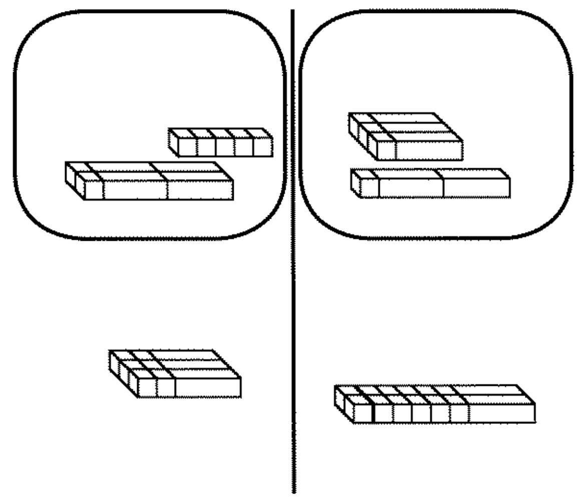
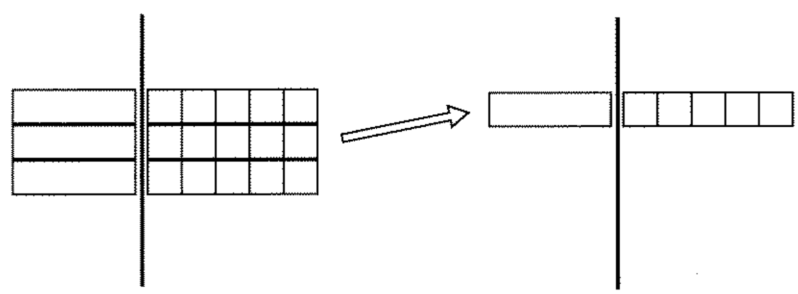


Another key to solving equations is the fact that you can *multiply or divide both sides by the same number* (as long as it's not zero).

For example, if $3x = 15$, then divide both sides by 3, and you find that $x = 5$.



Of course, some divisions cannot be shown easily with the blocks. If you end up with $4y = 7$, then dividing both sides by 4 will reveal that $y = \frac{7}{4}$. This is impossible to show with the Lab Gear.

Write and solve these equations.

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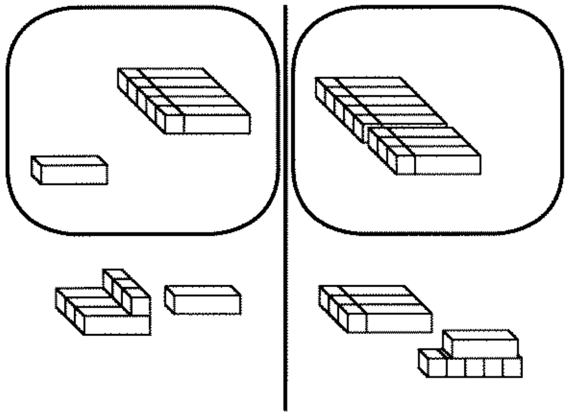

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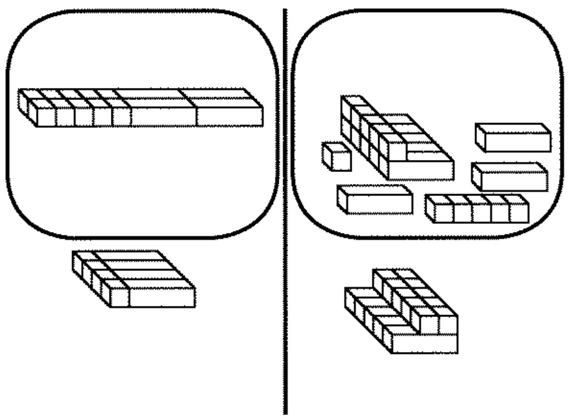

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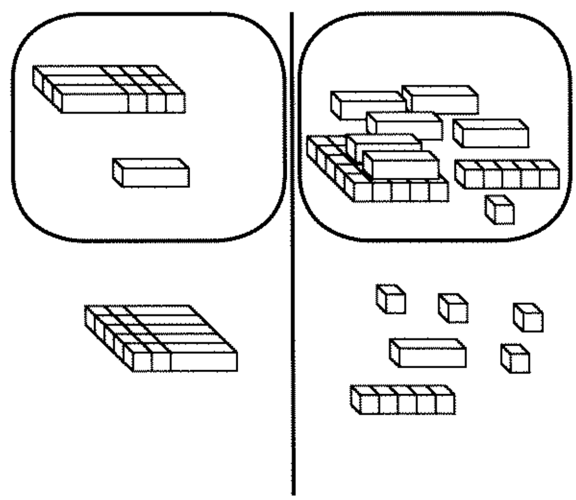

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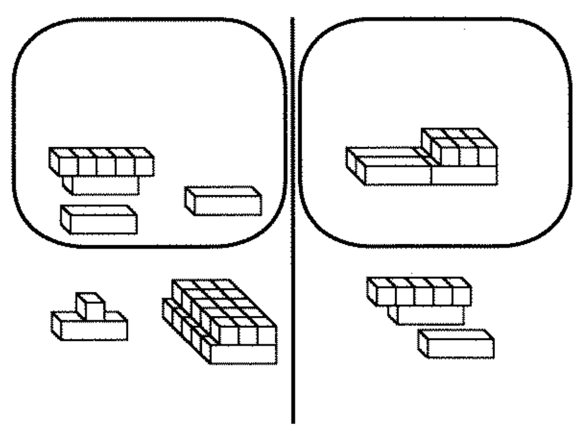

Solving Tricks

Many students discover tricks for solving equations with the Lab Gear. The figures show two simple equations. For each one, solve it, and explain any tricks or shortcuts that you used. Such tricks, if they are correct mathematically, can be used to simplify more complicated problems.

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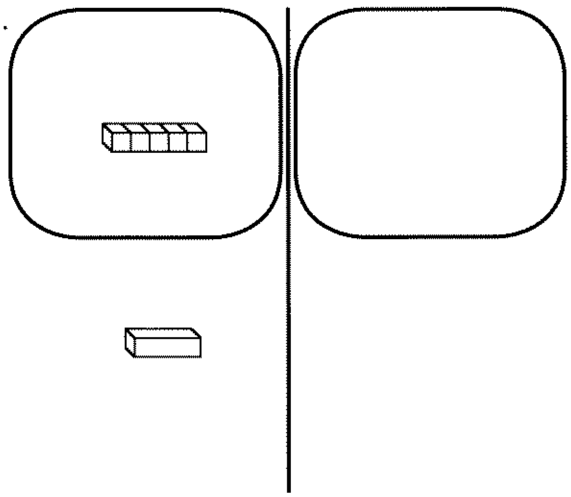

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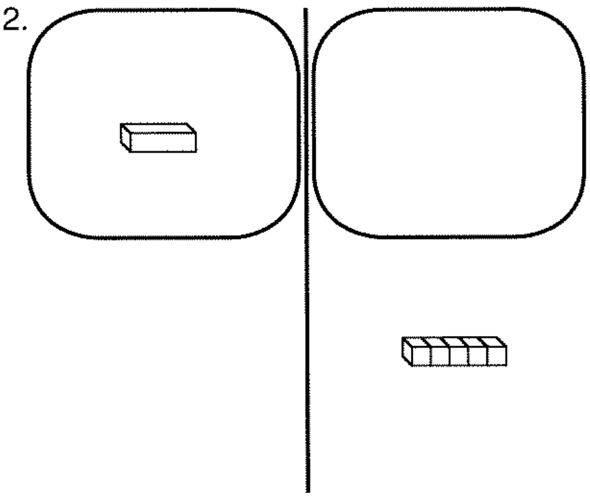

Use all the techniques and tricks you have learned so far to solve these equations. They may have no solutions, one solution, two solutions, or be identities.

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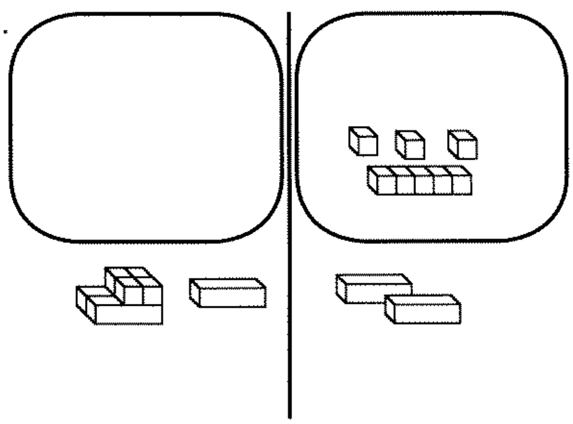

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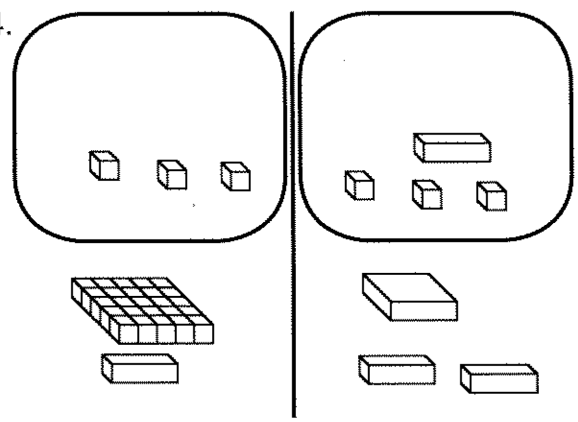

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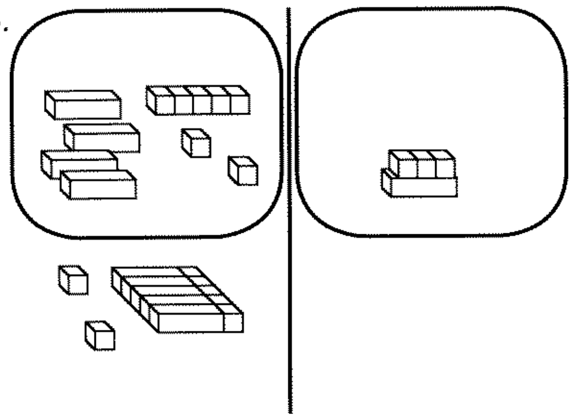

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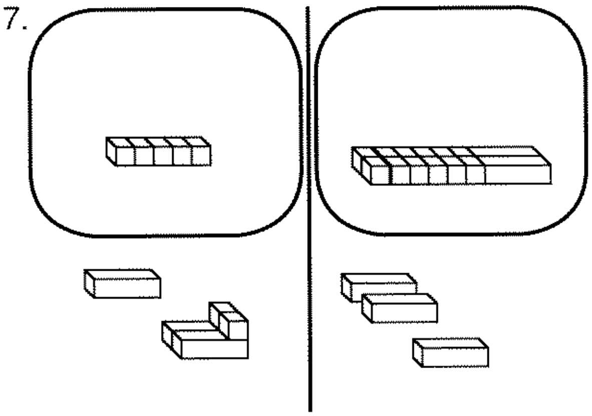

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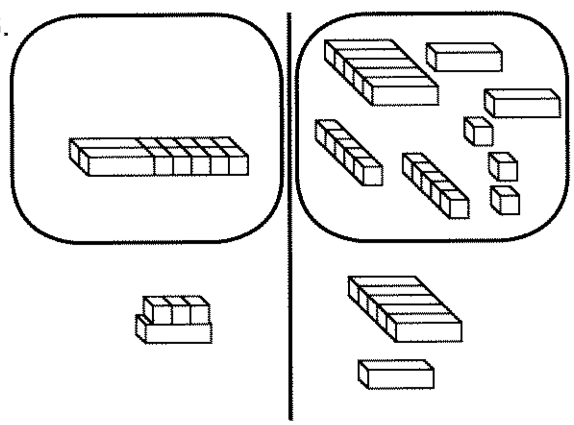

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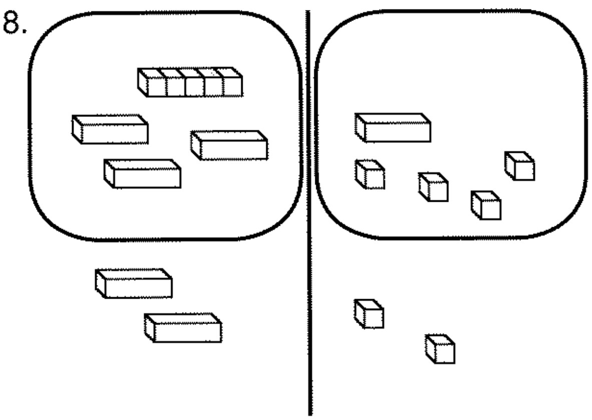

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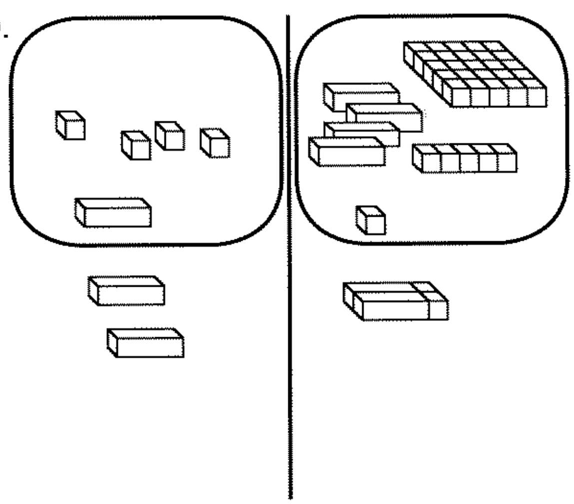

10. Were there any problems in the *How Many Solutions?* exploration at the beginning of this chapter that you couldn't solve? Try them again now.

Equations and Numbers

The first numbers people used were whole numbers. It took many centuries to discover more and more types of numbers. The discovery of new kinds of numbers is related to the attempt to solve more and more equations. The following equations are examples.

a. $x + 2 = 9$
b. $x + 9 = 2$
c. $2x = 6$
d. $6x = 2$
e. $x^2 = 9$
f. $x^2 = 10$
g. $x^2 = -9$

1. Pretend you only know about the **natural numbers**. (These are the positive whole numbers.) List the equations above that can be solved.
2. Pretend you only know about the **integers**. (These are positive and negative whole numbers, and zero.) List the equations above that can be solved. Find one that has two solutions.
3. Pretend you only know about the **rational numbers**. (These are all fractions, positive, negative, and zero. Of course, integers are included, since for example $3 = \frac{6}{2}$.) List the equations above that can be solved.

To solve equation (f), you need a number whose square is 10. The square root key of a calculator provides one answer: 3.16227766. But if you try to multiply this number by itself, you will find that the answer is not exactly 10.

Your calculator should say 9.999999999 for the product of 3.16227766 times 3.16227766. (A powerful computer, or an exceptionally patient and accurate student, would give the answer 9.999999998935076.) That means that 3.16227766 is very, very close to the square root of 10, but it is not exactly the square root of 10.

In later math classes, you will learn that the square root of 10 is not a rational number. It is an **irrational real number**. It is on the number line somewhere between the two rational numbers 3.16227766 and 3.16227767.

To solve equation (g), you need to get off the number line! The solution is a **complex number**, and it is written $3i$. The number i is a number one unit away from 0, but off the number line. $i^2 = -1$. You will learn more about i in an advanced algebra class.

Exploration 3 Make a Rectangle

1. Make as many different rectangles as you can with x^2, $10x$, and any number of yellow blocks. For each one, write a multiplication equation to show that area = length · width.

   Is one of the rectangles a square?

2. Make as many different rectangles as you can with x^2, 18, and any number of x-blocks. For each one, write a multiplication equation to show that area = length · width.

   Is one of the rectangles a square?

Exploration 4 Make a Square

To make a square with these blocks, add as many yellow blocks as you want, but nothing else. For each square, write an equation relating the side length to the area.

1. $x^2 + 10x + \ldots$
2. $4x^2 + 8x + \ldots$
3. $9x^2 + 6x + \ldots$
4. $x^2 + 2x + \ldots$
5. $4x^2 + 12x + \ldots$
6. Is it possible to get a different square by adding a different number of yellow blocks? Explain your answer.

To make a square with these blocks, add as many x-blocks as you want, but nothing else. For each square, write an equation relating the side length to the area.

7. $x^2 + \ldots + 25$
8. $4x^2 + \ldots + 25$
9. $x^2 + \ldots + 36$
10. $9x^2 + \ldots + 1$
11. $x^2 + \ldots + 9$
12. Is it possible to get a different square by adding a different number of x-blocks? Explain your answer.

Exploration 5 Perimeter

Look at each sequence. Think about how it continues, following the pattern. Write the perimeters of the figures given, then the perimeter of the fourth one, the tenth one, and the hundredth one.

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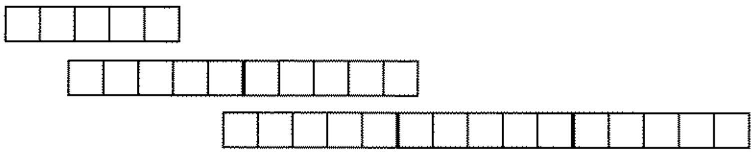
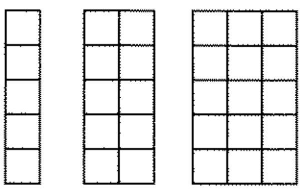
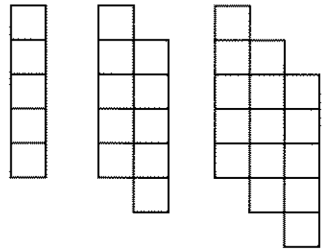
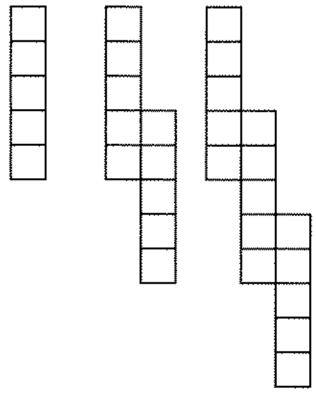
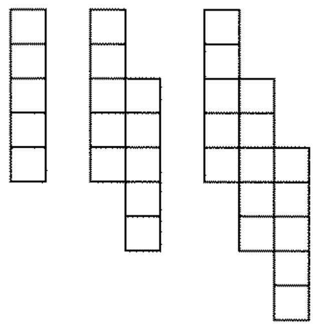
2. 

3. 

4. 

5. 

6. 

Exploration 6 Surface Area

Look at each sequence. Think about how it continues, following the pattern. Write the surface areas of the figures given, then the surface area of the fourth one, the tenth one, and the hundredth one.

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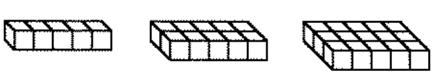
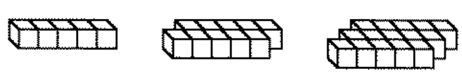
2. 

3. 

4.