Iterating Linear Functions

An Introduction to Dynamical Systems

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Introduction

The arrival of computers has caused some major changes in both mathematics and math education. One of the biggest shifts has been from an emphasis on symbolic methods to one on numerical methods. One field of mathematics, dynamical systems, requires considerable number crunching and is just coming into its own because we now have the ability to perform extensive calculations. The purpose of this article is to introduce students to this new field. The study of sequences created by using numerical iteration provides interesting new ways to approach many of the concepts central to the secondary mathematics curriculum, such as functions in general and linear and exponential functions in particular. Although the activities are calculator-dependent, students will still have to reason graphically, numerically and algebraically about functions. They will have to interpret graphs, find patterns in tables of data, and use algebra both to simplify expressions and to derive new meanings from them. In addition, they will be given some new modeling tools and will be using what they learn in an applications setting. Finally, these activities will give students the necessary background to later explore the exciting new mathematics of chaos and fractal geometry.

Prerequisites: basic algebra

Grade Level: 9-12

Materials: The activities require a lot of computation and can be done using any calculator, but one with an ANS key would be preferable. A spreadsheet program could also be used.

Objectives: These activities will enable students to learn the basics of numerical iteration and then use it as a tool for creating mathematical models.

Teacher's Guide

Sheet 1:

This sheet introduces the basic concepts, vocabulary and tools for this activity.

Use the birthday problem to establish how students will perform the calculations. This depends on what technology they have access to. If they have calculators with an ANS key, here is how to generate orbits, using the example given on the student sheet:

- ◊ Clear the screen.
- ◊ Enter 24 in your display and press ENTER.
- ◊ Type in (1/2)*ANS + 4, press ENTER and 16 should appear in the display.
- ◊ Now, press ENTER again and 12 will appear.
- ◊ Each time you press ENTER you get the next term in the sequence.

Other methods are possible, depending on the capabilities of the calculators or software your students have access to.

Ask students to compare their results with neighbors. They should find that all orbits get closer and closer to 8. (If some students don't, help them check their calculation technique!)

Demonstrate how to fill out the table and graph for #1 and #3. Note that students can share the computation with their neighbors, but everyone must end with a complete record. For #2 and #4, students will need copies of the Time Series Record Sheet. They will have to make decisions on how to set up the scale for the graphs, which is easier to do well after having computed a few orbits.

The time series graph showing several orbits at once is an application and refinement of the function diagram and linked function diagrams representations (Wah and Picciotto, 1994).

Some students may notice that the time series graphs look like exponential curves, and indeed they are. See the **Mathematical Background** section for an explanation.

For #5, accept any reasonable answer, including a visual one based on looking at the graphs and guessing. However make sure students check that the iterates for the seed they suggest as having a horizontal graph indeed remain fixed. Some students may notice that in the first example, the time series graphs all seem to get closer and closer to the fixed point, while in the second example they seem to move away from it. They will be asked to do a more thorough analysis of the fixed point on Sheet 4.

Sheet 2.

To verify understanding of #1-2, you may discuss what the formula would be with different values for the percent eliminated per hour, or for the hourly dose. Also ask the students why the initial dose of 100 units does not show up in the formula (Answer: it is actually a seed, not a part of the equation.)

The answer to #4 depends on how you interpret the question. Does the level of FluRidder need to pass 80 units some of the time, does it need to never drop below 80 units? When going over this with your students, you may make a table that shows the level of medicine both immediately before and immediately after taking the hourly dose. Figure 1 shows a detailed analysis of the problem:

```
          initial dose            first hour
  0.00 ------------->  20.00 ------------->
            dose #1               second hour
 13.60 ------------->  53.60 ------------->
            dose #2               third hour
 36.45 ------------->  76.45 ------------->
             dose #3               fourth hour
 51.99 ------------->  91.99 ------------->
             dose #4               fifth hour
 62.55 -------------> 102.55 ------------->
             dose #5               sixth hour
 69.73 -------------> 109.73 ------------->
             dose #6              seventh hour
 74.62 -------------> 114.62 ------------->
             dose #7               eighth hour
 77.94 -------------> 117.94 ------------->

 80.20
```

Figure 1: units of FluRidder in the body

The level of FluRidder immediately after taking the initial dose is 20 units. Then the level drops gradually, until it is down to 0.68 * 20 = 13.6 units immediately before taking the next dose. Immediately after taking that dose, the level is 13.6 + 40 = 53.6 units, which is the number given to us by the formula. After two more iterations, the formula give us 91.99 units, which is greater than 80, and satisfies the first condition. Three hours have passed.

However, at some point during the following hour, the level will drop below 80 units, on its way to a low of 0.68 * 91.99 = 62.55 units. To make sure the level never drops below 80 units, you must wait for the amount in your system to reach or pass 117.65 units right after taking the medicine, which would guarantee you had at least 80 units in your system right before taking the hourly dose, since 0.68 * 117.65 = 80.00. This happens at the end seven hours, when the amount of FluRidder in the system is 117.94 units.

Note that an alternate formula to think about this problem (as suggested in #6) is

$$y = 0.68 (x + 40)$$

In this case, the x represents the amount of medicine in the system immediately prior to taking the hourly dose, and the y the amount immediately prior to taking the next dose. The seed for #4 would be 13.6 units, the amount immediately prior to taking the first 40-unit dose. If you iterate this formula, you will get the numbers in the first column of the table in Figure 1, and find that the amount of medication in your system before taking the hourly dose stabilizes around 85 units. On the other hand, iterating the formula from #2 shows that the amount of FluRidder in the system immediately after taking the hourly dose (the number in the second column). This one stabilizes around 125 units. Not surprisingly, this is 40 units higher than 85 units. Note that 40 is 32% of 125, as it should be.

It is not necessary for students to fully understand this right away. Depending on the class, you can have a full discussion of the question, and analyze what happens with each of the two formulas, or a briefer discussion of the situation being modeled, stressing what part of it is represented by the formula in #2.

Since most of the FluRidder is purged from your body during the night, it turns out that you get the same answer whether you do #5 by carrying out the iteration over three days, or just start at 6 a.m. that day. Again, you may decide how deeply to get into this depending on the level of the class.

Sheet 3.

These applications to financial matters work the same way, though they may be a little easier to understand since there is no continuous change as in the medication problem. However, students may need help figuring out how to use negative numbers to describe the loan in #4.

In #4, students need to choose both the seeds and the scale of the axis. It is imperative that the range from -1000 to 1000 be displayed, since that is what the bicycle problem requires. However, to find the horizontal line in the graph, the lower bound of the scale needs to include -7500 (see Sheet 4, #1b.)

#5 is challenging. Trial and error can quickly narrow the search to an approximate range, but to get an exact answer it is best to work backwards: if the amount in the account ends at $0, it had $50,000 in it immediately preceding the last payment, and $50,000/1.1 a year before that. More generally, if x is in the account at a given time, a year earlier the amount was $y = (x+50{,}000)/1.1$. This is the inverse function to the one mentioned in the problem. To find the initial deposit, iterate this function nineteen times, with a seed of 0.

The general linear function that gets iterated for many financial accounts is $y = (1 + r)x + P$ where r is the interest rate, and P is the amount of money deposited or withdrawn. With a seed of S, here is a summary of the types of accounts:

- If $S > 0$ and $P = 0$, then the account is a *savings account*.
- If $S > 0$ and $P > 0$, then the account is an *annuity* or *retirement fund*.
- If $S > 0$ and $P < 0$ then the account is called a *sinking fund*.
- If $S < 0$ and $P > 0$ then the account is called a *loan* or a *mortgage*.

Sheet 4.

If #4a is too difficult for your class, you may demonstrate the solution.

Students at all levels should see that the value of m in $f(x) = mx + b$ determines the type of fixed point. It would be appropriate for upper level students to prove their conjectures. This can be done in a variety of ways, one of which is outlined in the **Mathematical Background** section, below.

One way to test whether the time series graphs lines move towards or away from each other is to test the first iterates of seeds 0 and 1: $y = m(0) + b = b$, and $y = m(1) + b = m + b$. The difference between the seeds was 1, and the difference between the first iterates is m. It follows that if $m > 1$, the lines move away from each other (repelling fixed point) and if $0 < m < 1$ the lines move towards each other (attracting fixed point.) More generally, if you take two seeds x_1 and x_2 and the corresponding first iterates y_1 and y_2, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ indicates whether the lines move away from or towards each other . Of course, that ratio is m, the rate of change of the function, which we usually think of as the slope of the line in a Cartesian graph.

Mathematical Background

If you are not familiar with this material, start by working through the student sheets. Then, for more information, read through this section. In order to allow access to the student sheets for the broadest possible range of classes, we avoided function and subscript notation there. However we will use both in this section.

We will label the seed x_0, and say that the sequence $x_0, x_1, x_2, x_3, \ldots, x_k$, is the orbit of x_0 under iteration of f(x). Note that $x_1 = f(x_0)$, $x_2 = f(x_1)$ etc., and that in general $x_k = f(x_{k-1})$. In other words, the next number in the orbit is calculated by applying the function rule to the preceding number. *Time series graphs* are produced by plotting the points $(0, x_0)$, $(1, x_1)$, $(2, x_2)$, . . . , and then connecting each point to its predecessor with a line segment. The line is not really part of the graph but shows how the points are related to each other.

By definition, f(x) has a fixed point at x = F if f(F) = F. By solving the equation x = mx + b for x, it can be shown that the linear function f(x) = mx + b has a fixed point at x = F = b/(1 - m). Once you have found the fixed point for a function, the next question of interest is what kind of fixed point is it, attracting or repelling? A little algebra throws light on this. Since b = (1 - m)F, the function can be written: f(x) = mx + F - Fm, or f(x) - F = m(x - F). Therefore the relationship between consecutive iterates is:

$$x_k - F = m(x_{k-1} - F)$$

This means that the directed distance between an iterate and the fixed point equals m times the directed distance between the previous iterate and the fixed point. This has interesting consequences:

- ◊ the distance between iterates and the fixed point is a geometric sequence, which explains the exponential shape of the time series graphs.
- ◊ if $|m| < 1$, the distance decreases, we have an attracting fixed point, and the orbits decay towards it exponentially.
- ◊ if $|m| > 1$, the distance increases, we have a repelling fixed point, and the orbits grow away from it exponentially.

Assessment

#9 in Sheet 4 provides an opportunity to evaluate student understanding. Expect a two-page illustrated summary of the key ideas, including examples. You may offer extra credit for a clear explanation of the “inverse” problem (see the lottery problem: Sheet 3, #6.)

Extensions

Exponential growth and decay. In both modeling exercises we had examples of exponential behavior: decay in the case of how your body metabolizes medication and growth in the case of financial accounts. There are many interesting questions which they can now pursue on their own. Here are a few:

- How does your body process different types of medication? When doctors and pharmacists talk about the half-life of a drug, what do they mean?
- How does your body process alcohol, or caffeine?
- What actual interest rates do banks use for the different types of accounts and why? what are the consequences for consumers?
- How do credit cards and debit cards work?

Algebraic analysis. If students have studied sequences and series and composition of functions, they can explore the ideas in the above **Mathematical Background**. An alternate approach is to observe that one can iterate f(x) = mx + b in the general case and produce the sequence:

$$x,\ mx + b,\ m(mx+b) + b = m^2x + mb + b, \ldots$$

Using the formula for the sums of a finite geometric series, a formula for the k^{th} iterate can be derived and used to analyze the behavior of the orbits.

More on dynamical systems. For more on dynamical systems, students can look at Devaney (1989), McGuire (1990), and Sandefur (1993). The TI-82, -83, and -92 calculators have a built-in

SEQUENCE mode which allows the user to quickly generate orbits in both table form. Moreover they support so-called “web” diagrams of the iteration process. These features make it possible to explore the iteration of non-linear functions, such as quadratics and trig functions.

In this article, we have dealt with linear functions of the form f(x) = mx + b where x, m and b are real numbers. All the theory we have developed also holds for linear functions in general. In particular, they hold for f(z) = $\alpha z + \beta$, where z, α, and β are complex numbers, and

$$F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}.$$

Those interested in transformation geometry can study the iteration of these types of linear functions. The results are a spectacular extension of the work done here (Heid et al, 1995.)

Selected Answers

Sheet 1

5. First graph: horizontal line for the seed 8. Second graph: horizontal line for the seed 4.

Sheet 2

1. a. 68%
   b. .68x

2. x is the amount of FluRidder in your body at a given time, right after taking the medication. 68x is the amount left in the body an hour later. 40 is the number of units in the hourly dose. y is the amount of the medication in your body right after taking the medication an hour after the given time.

4. 3 hours to reach 80 units, 7 hours to never drop below 80 units.

5. 2 p.m. on Tuesday.

6. See teacher’s guide.

Sheet 3

1. The x stands for the amount of money in the account immediately after a given deposit. .01x is the interest earned by the end of the month. 75 is the dollar amount of the monthly deposit. The y stands for the amount of money in the account after the deposit a month after the given deposit.

2. 12 months

3. 13 months. The total cost would be 100+13*75 minus the amount of money in the account at that time, which is $28.49, or $1046.51.

4. Answers will vary.

5. a. 1.1x represents the amount in the account after the interest has been received. The -50,000 is the amount paid out to the winner each year.
   b. $418,246.01 (See Teacher’s Guide.)

6. In this case, x is the amount of money immediately before a given deposit is made, and y is the amount of money in the account immediately before the next deposit is made.

Sheet 4.

1. a. 125 units.
   b. $7500.00
2. A horizontal line.
3. a. 100

b. 6.8888…

4. a. $\frac{-b}{m-1}$
5. The orbits approach 125.
6. The orbits move away from -7500.
7. See Teachers' Guide.
8. y = .9x + 10 has lines that move towards each other, and an attracting fixed point.
   y = 2.8x - 12.4 has lines that move away from each other, and a repelling fixed point.

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