

# Infinity

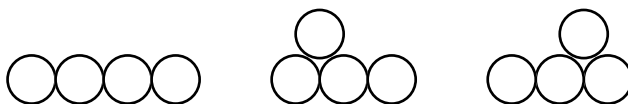
## **Unit 3: Stairway to Infinity! Proof by Mathematical Induction**

## The Strong Law of Small Numbers

Richard K. Guy, a contemporary mathematician in Calgary, Canada, formulated the Strong Law of Small Numbers: *You can't tell by looking – coincidences cause careless conjectures!*

Guy collects examples of patterns that work with small numbers. However, many of them break down sooner or later. Here are a few examples. Some break down, some don't. For each one, find the pattern, and try to figure out if it keeps going or breaks down.

1. What is the greatest number of pieces you can get by making  $n$  straight cuts through a circular pizza?
2. What is the greatest number of pieces you can get by joining  $n$  points on a circle in every possible way?
3. What is  $\frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}$  for  $n = 1, 2, \dots$
4. How many ways are there to arrange  $n$  pennies flat on a table in unbroken rows so that each penny not on the bottom row touches two pennies below it. For example, there are three ways to arrange four pennies following these rules:



5. Is  $n^2 + n + 41$  always prime for  $n = 1, 2, 3, \dots$ ?
6. Can every odd number greater than 3 be written as a sum of a prime and a power of 2?

**Definition:**  $n!$  (read “ $n$  factorial”) is the product  $1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ . For example,  $3! = 6$ , and  $4! = 24$ .

7. A number is a Niven number if it is divisible by the sum of its digits. For example, 21 is a Niven number, since it's divisible by 3, but 22 is not since it's not divisible by 4. Check whether  $n!$  is a Niven number for  $n = 1, 2, 3$ , etc.
8. Can every odd number greater than 1 be written as a prime plus two times a square?
9. **Goldbach's Conjecture:** Every even number greater than 2 can be written as the sum of two primes.

## Introduction to Mathematical Induction

*Mathematical induction* is used to prove that a fact is true for *all* (natural number) values of  $n$ .

- ◇ Proving the fact works for the initial term (or terms) of the sequence is *the anchor*.
- ◇ Proving that *if it works for  $n-1$ , then it works for  $n$*  is the *inductive* (or recursive) *step*.
- ◇ If you prove both, you are done.

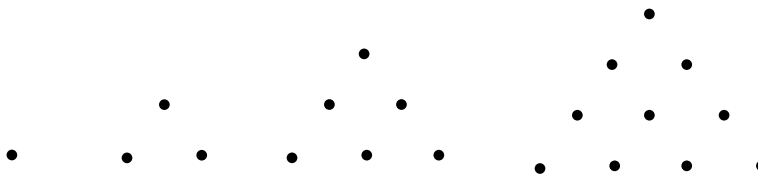
One way to help you prove a formula by mathematical induction uses the understanding you already have of recursive and explicit formulas:

*Recursive Formula:* equation expressing  $x_n$  in terms of  $x_{n-1}$  (and sometimes  $x_{n-2}$ , and so on.)

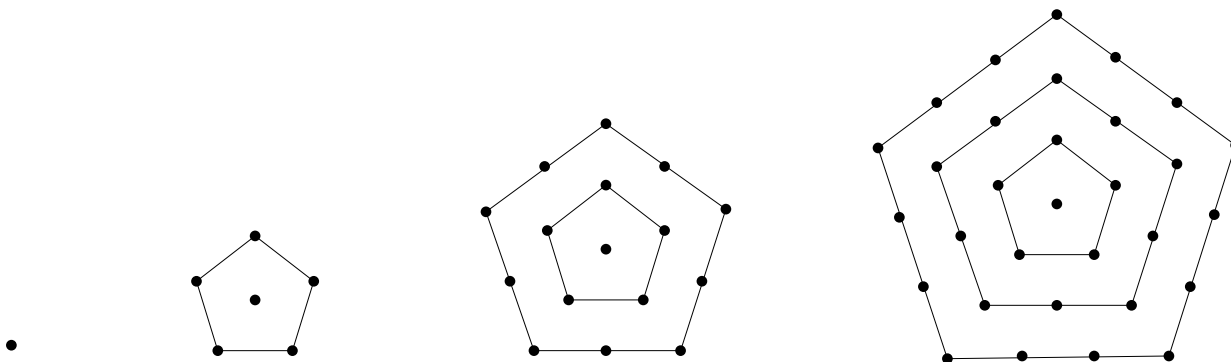
*Explicit Formula:* equation expressing  $x_n$  in terms of  $n$ .

For each problem below:

- a. List the first eight terms of the sequence.
- b. Make a conjecture about the explicit formula for  $x_n$ .
- c. Write the same formula for  $x_{n-1}$ .
- d. Find the recursive formula.
- e. Use algebra to derive the equation you found in (b) from the ones you wrote in (c) and (d).
- f. You are now ready to write a proof by mathematical induction! Do it.



1. The  $n^{\text{th}}$  triangular number.
2. The  $n^{\text{th}}$  pentagonal number .



3. The  $n^{\text{th}}$  odd number.
4. The sum of the first  $n$  odd numbers.
5.  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n \cdot (n + 1)}$
6.  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$
7. The sum of the first  $n$  square numbers.
8. The sum of the first  $n$  triangular numbers.
9. The sum of the first  $n$  cubes.
10. The number of rectangles of any size (including squares) in an  $n$ -by- $n$  checkerboard.

## Fibonacci Conjectures

Make some conjectures about Fibonacci and Lucas numbers, and then try to prove them. The following are hints. Other conjectures, of course, are possible.

$$L_n - F_n$$

$$L_n \cdot F_n$$

$$F_n \cdot F_{n+2}$$

$$L_n \cdot L_{n+2}$$

Sum of every other Fibonacci number (even subscripts)

Sum of every other Fibonacci number (odd subscripts)

Sum of first n Lucas numbers

Sum of every other Lucas number (even subscripts)

Sum of every other Lucas number (odd subscripts)

$$F_n^2 + F_{n+1}^2$$

$$L_n^2 + L_{n+1}^2$$

$$F_{n+1}^2 - F_{n-1}^2$$

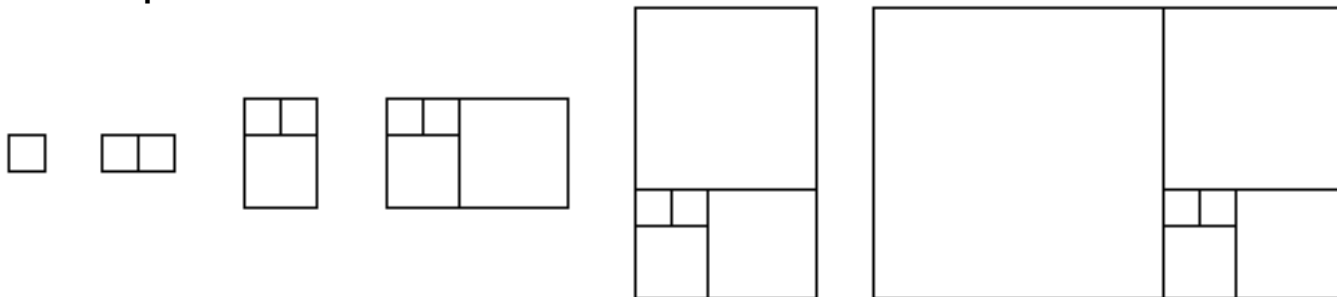
$$F_n + F_{n+k}$$

$$F_n - F_{n+k}$$

Sum of squares of first n Fibonacci numbers

## Visual Fibonacci

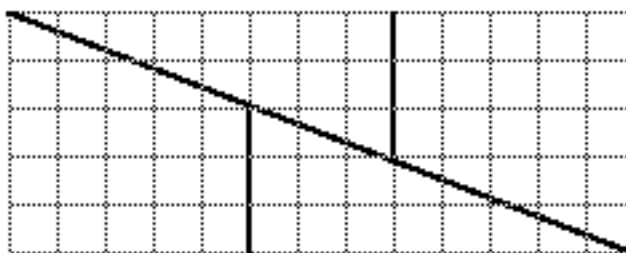
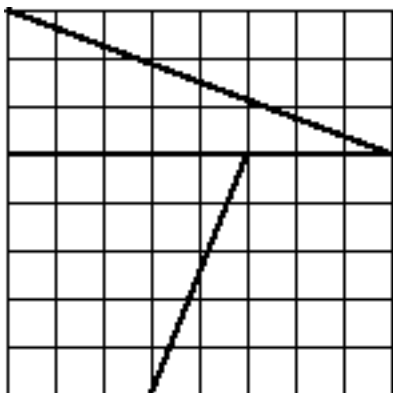
### Golden Spiral



1. The figure above illustrates a Fibonacci formula. (All the component shapes are squares.) What is the formula?
2. Prove the formula.
3. Use the last drawing in the series to draw a spiral by a succession of quarter circles.
4. Add an extra square in order to continue the spiral.


### Impossible!



Lewis Carroll liked this puzzle:

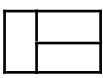
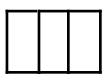
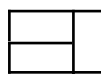


5. In the figure above, what is the area of the square?
6. The pieces of the square can be rearranged to make the figure on the right. What is its area?
7. How do you explain the apparent contradiction?
8. Use Fibonacci numbers to generalize this puzzle, using Fibonacci numbers for the square and rectangle dimensions. Sketch examples on graph paper.
9. Is the trick more or less obvious with larger versions of the puzzle?
10. What is the Fibonacci formula that is involved in this puzzle? Can you prove it?

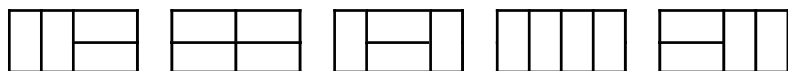
### Dominoes and Fibonacci Numbers

There is one way to tile a 2 by 1 strip with a domino: 

There are two ways to tile a 2 by 2 strip with dominoes:  

There are three ways to tile a 2 by 3 strip with dominoes:   

There are five ways to tile a 2 by 4 strip with dominoes:



1. There are  $F_{n+1}$  to tile a 2 by  $n$  strip with dominoes. Prove this by mathematical induction. (Hint: a 2 by  $n$  strip can be created only by adding a vertical domino to a 2 by  $n-1$  strip, or by adding two horizontal dominoes to a 2 by  $n-2$  strip.)

We will use this fact to prove the conjecture about the sum of consecutive squares of Fibonacci numbers:  $F_n^2 + F_{n+1}^2 = F_{2n+1}$ .

Take  $n = 5$  as an example. We will use domino tilings to show that  $F_5^2 + F_6^2 = F_{11}$ .  $F_{11}$  is the number of tilings for a 2 by 10 strip. What is at the exact center of the strip? Either it breaks into two equal strips of length 5, or it breaks into a strip of length 4, two horizontal dominoes, and another strip of length 4:



The first figure can be tiled in  $F_6 \cdot F_6$  ways, since we are combining the ways to tile the first half with the ways to tile the second half. Likewise, the second figure can be tiled in  $F_5 \cdot F_5$  ways. Since there are no other possibilities to tile the 2 by 10, we have shown that  $F_5^2 + F_6^2 = F_{11}$ . QED.

2. Write this proof in the general case, to show that the conjecture is true for any  $n$ .

The Fibonacci conjecture about the “Impossible!” puzzle can be proved using dominoes, but it is more difficult.

## An Explicit Formula for the Fibonacci Sequence

The notation  $\lceil x \rceil$  stands for the ceiling function, which outputs the smallest integer greater than or equal to  $x$ . For example,  $\lceil 2.4 \rceil = 3$

1. What is  $\left\lceil e^{\frac{n-2}{2}} \right\rceil$  for  $n = 1, 2, 3, \dots$ ? Discuss this with your neighbors.

This is the Fibonacci sequence:  $F_1=1, F_2=1, F_n=F_{n-1}+F_{n-2}$

2. Is the Fibonacci sequence “more like” an arithmetic sequence, or more like a geometric sequence? Do many calculations to back up your claim. (Hint: do not forget to consider large values of  $n$ .)

Any sequence created by using the same recursive formula is called a generalized Fibonacci sequence. For example, the Lucas sequence ( $L_1=1, L_2=3, L_n=L_{n-1}+L_{n-2}$ ) is a generalized Fibonacci sequence.

3. Find a sequence that is both geometric and a generalized Fibonacci sequence. (Hint: use  $G_0=1$  and  $G_1=x$ . Find a value of  $x$  that makes this possible.)

If you did this correctly, you actually found two values of  $x$  that work. Call the negative value  $\sigma$  (sigma), and the positive one  $\tau$  (tau). So we have two geometric / generalized Fibonacci sequences:  $1, \tau, \tau^2, \dots$  and  $1, \sigma, \sigma^2, \dots$ . Having found these two sequences will make it possible to get an explicit formula for the Fibonacci sequence itself.

4. Find the approximate decimal values of  $\tau$  and  $\sigma$ . One of these numbers is related to question #2 above. How?

In the rest of this sheet, use only the *exact* values of  $\tau$  and  $\sigma$ .

5. Explain why we have:

a. $1+\sigma=\sigma^2, 1+\tau=\tau^2$	b. $\sigma^n=\sigma^{n-1}+\sigma^{n-2}, \tau^n=\tau^{n-1}+\tau^{n-2}$
---------------------------------------	---

6. Simplify:

a. $\tau-\sigma=$	b. $\tau+\sigma=$	c. $\tau\sigma=$	d. $1+1/\tau=$
-------------------	-------------------	------------------	----------------

7. Calculate the exact values of the first terms of the sequence  $1, \tau, \tau^2, \dots$  (Do not use decimal approximations.)

8. Generalize for  $\tau^n$ .

9. Prove your generalization.

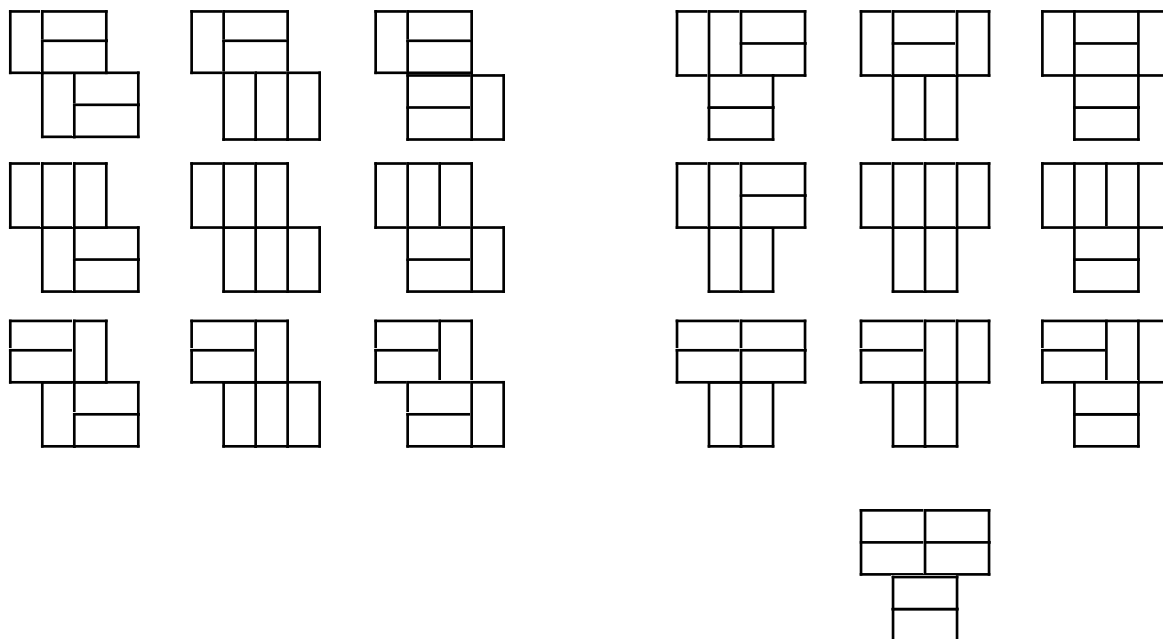
10. Repeat #6-8 for  $\sigma^n$ .

11. Use your formulas for  $\sigma^n$  and  $\tau^n$  to find explicit functions for the Fibonacci and Lucas numbers.

The explicit formulas provide another way to prove Fibonacci conjectures.

## Squares of Fibonacci Numbers

### Using Dominoes



### Using the Explicit Formula

Use the explicit formula to find  $F_{n-1} \cdot F_{n+1}$  in terms of  $\sigma$  and  $\tau$ . Use properties of  $\sigma$  and  $\tau$  to simplify.