

# Infinity

## Unit 2: Chaos! Dynamical Systems

## Iterating Linear Functions

These questions are about iterating  $f(x) = mx + b$ . Seed:  $x_1$ . Orbit:  $x_1, x_2, x_3, \dots$

**For each question, give examples and a symbolic answer.**

1. What values of  $m$  and  $b$  yield
  - a. an arithmetic sequence?
  - b. a geometric sequence?
2. What happens if
  - a.  $m = 0$ ?
  - b.  $m = -1$ ?

A fixed point is a value of  $x_1$  such that its orbit is  $x_1, x_1, x_1, \dots$

3. When iterating a linear function,
  - a. if there is a fixed point, find a formula for it
  - b. when does it not exist?
4. For what values of  $m$  and  $b$  is the fixed point
  - a. attracting? (all orbits have the fixed point as a limit)
  - b. repelling? (all orbits move away from the fixed point, towards infinity)
5. Find the differences ( $d_1, d_2, d_3, \dots$ ) between successive iterates and the fixed point. What sort of sequence do you get? (arithmetic? if so, what is the common difference? geometric? if so, what is the common ratio? neither? Try this at least once with  $m < 1$ , and once with  $m > 1$ .)
6. Find an explicit formula for  $d_n$ .
7. Find an explicit formula for  $x_n$ .

### Graphical Iteration

1. Match the values of  $m$  to the appropriate figure.

a.  $m < -1$

b.  $m = -1$

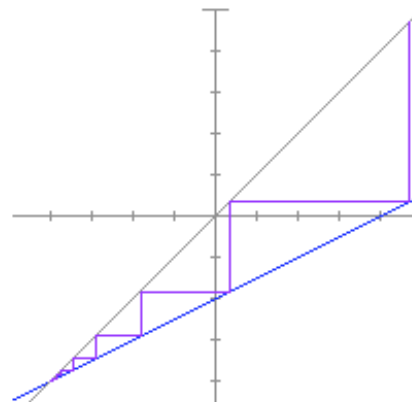
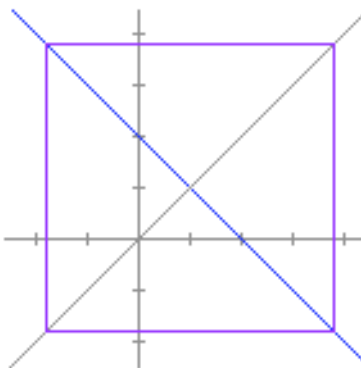
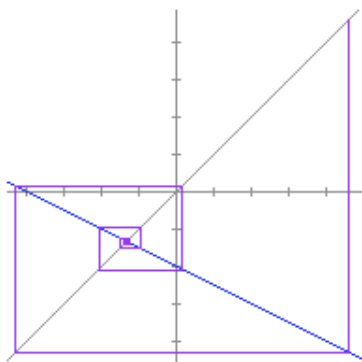
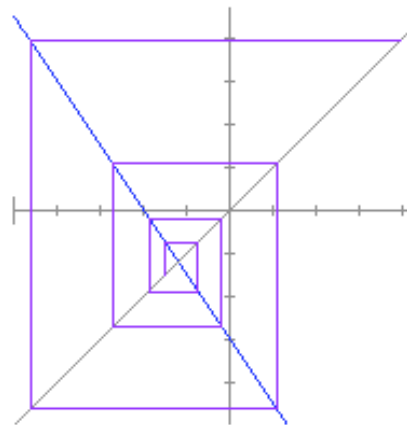
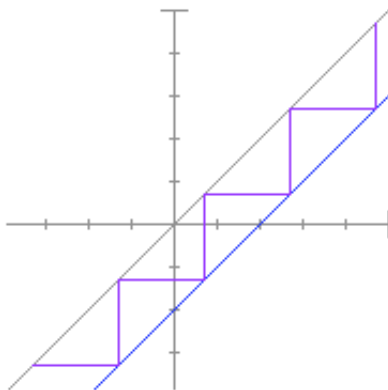
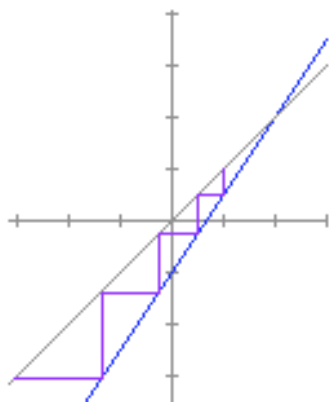
c.  $-1 < m < 0$

d.  $0 < m < 1$

e.  $m = 1$

f.  $m > 1$

2. Some orbits are attracted to the fixed points, some are repelled, one is periodic. Which is which?



## Orbit Analysis

(Examples from Robert Devaney's *Chaos, Fractals, and Dynamical Systems*, ch 1.)

When iterating  $f(x)$ , eight different kinds of orbits are possible:

- An orbit may approach positive or negative infinity
- An orbit may be fixed
- An orbit may be eventually fixed
- An orbit may approach a fixed point
- An orbit may be periodic
- An orbit may be eventually periodic
- An orbit may approach a period
- An orbit may be chaotic

1. What is the orbit of 0 for the following functions?

- $f(x) = |x|$
- $f(x) = 1 - x^2$
- $f(x) = (x - 1)^2$
- $f(x) = x^2 - 2$
- $f(x) = x^2 - 2x - 1$
- $f(x) = \frac{(x - 2)(3x + 1)}{-2}$

2. Analyze the orbits for the following functions, for different seeds:

- $f(x) = \sqrt{x}$
- $f(x) = x^2$
- $f(x) = x^2 - 2$
- $f(x) = x^3$
- $f(x) = -x^2$
- $f(x) = 1/x$
- $f(x) = -x^3$
- $f(x) = (x + 1)(-1.5x + 1)$
- $f(x) = 1/x^2$
- $f(x) = 1/\sqrt{x}$

### Iterating the Logistic Equation

$$y=rx(1-x) \text{ with } r>0$$

We will do an in-depth orbit analysis of the above equation, for different values of r.

1. Explain why a value of  $x_n > 1$  leads to a negative value of  $x_{n+1}$ . What happens with a negative value?
2. How many fixed points are there? Find a formula for each, as a function of r if necessary.
3. For what values of r is the entire curve under the  $y=x$  line in the domain  $[0,1]$ ?
4. Where is the vertex? Find a formula for its coordinates as a function of r.
5. For what values of r is the vertex outside the unit square?
6. What are the implications for iterating the function in those cases?
7. Do an orbit analysis for the logistic equation. Consider only seeds in the interval  $[0,1]$ . See how much you can answer by studying the figures below, and then use technology. Consider the following cases:
 

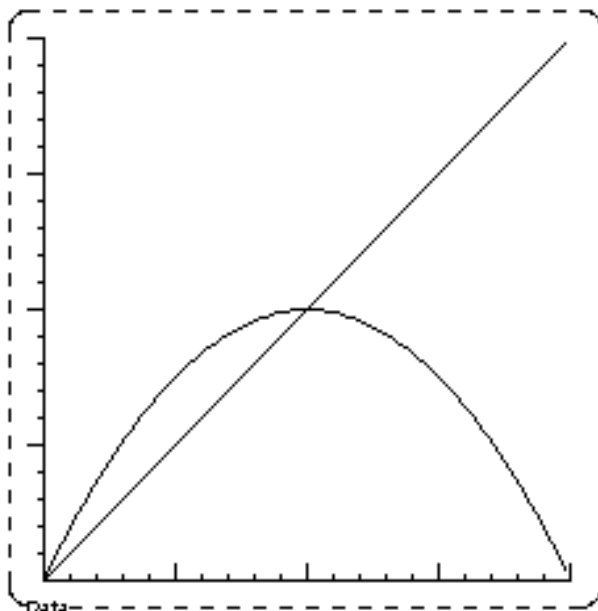
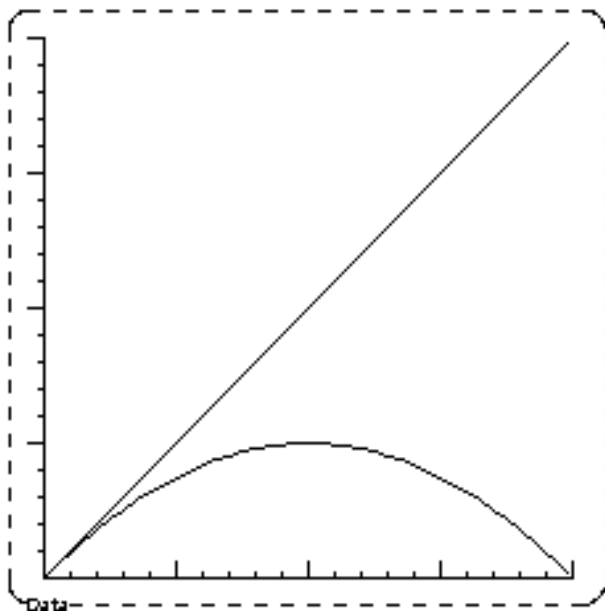
a. $0 < r < 1$	b. $r = 1$	c. $1 < r < 2$
d. $r = 2$	e. $2 < r < 3$	f. $r = 3$
g. $3 < r < 4$	h. $r = 4$	i. $r > 4$

#### Graphs with Various Values of r

**Unit square:**  $x_{\min} = 0, x_{\max} = 1, y_{\min} = 0, y_{\max} = 1$ .

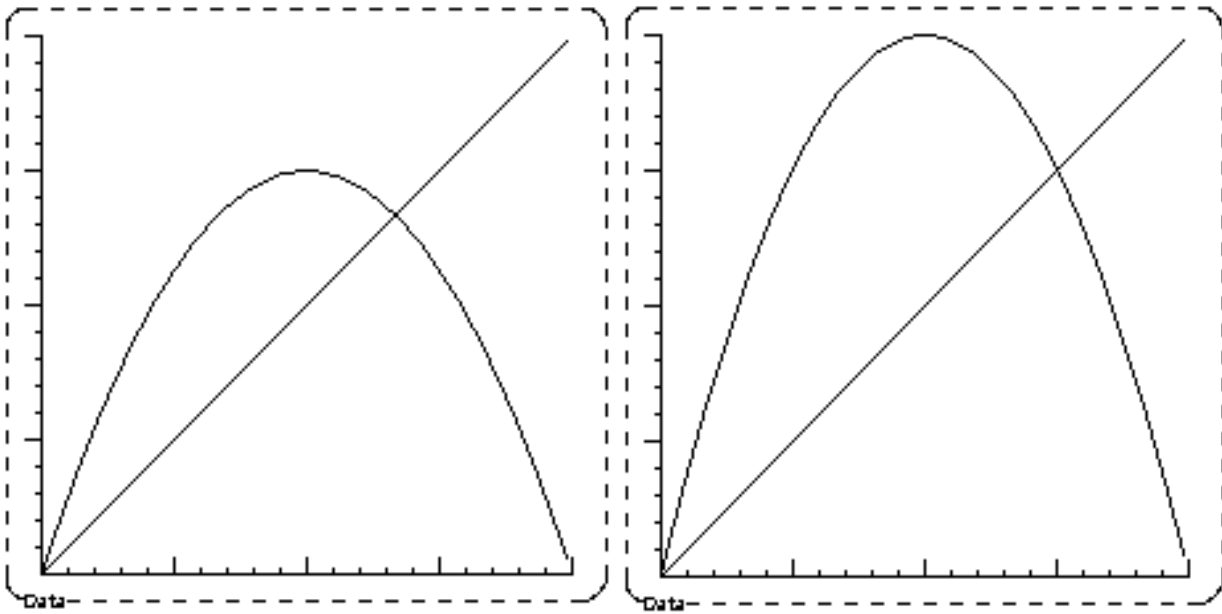
r=1

r=2

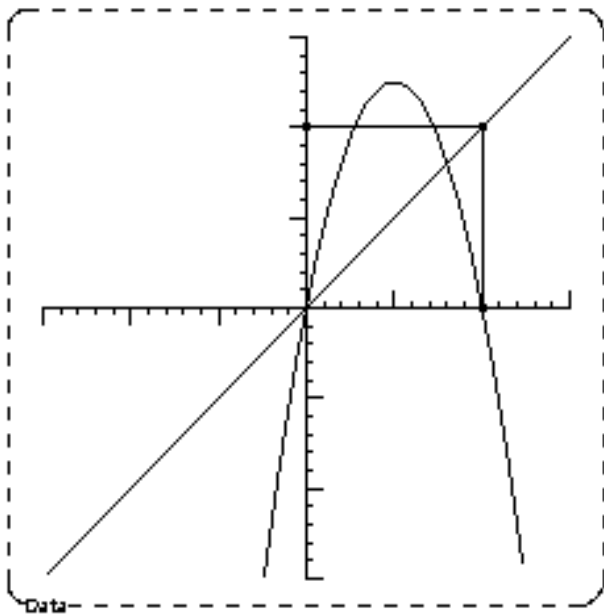


r=3

r=4



$r=5$  (this is zoomed out a bit, the unit square is marked)



## Building a Bifurcation Graph

You will use Fathom to build the bifurcation graph for the logistic function.

1. Create a new Fathom file, or edit a file you already have. The file should have:
  - a **table** with columns for  $n$  and  $x$
  - a **graph** of  $x$  as a function of  $n$
  - a **slider** for the seed (0 to 1)
  - a **slider** for  $r$  (0 to 4)
  - a small **function plot** graph of  $y=x$  and the logistic function
2. Double-click the collection icon (a box filled with golden balls), and click on the **Measures** tab. Create two measures:
  - a. **parameter**, with formula  $r$ .
  - b. **endpoint**, with formula  $\text{last}(x)$
  - c. make sure you understand the meaning of these two measures, because if you don't it doesn't make sense to go on
3. Select the collection icon, and choose **Collect Measures** from the **Collection** menu. This creates a new collection, called **Measures from....** Double-click it, and select the **Collect Measures** tab.
  - a. turn off animation (which slows things way down)
  - b. uncheck **Replace existing cases** (this will help you collect many values for  $\text{last}(x)$  as  $r$  changes)
  - c. check **Re-collect measures when source changes**
  - d. we only need **1** measure for each value of  $r$
4. Make a new graph, with **parameter** on the x-axis, and **endpoint** on the y-axis.
5. Animate both sliders. You'll see a bifurcation graph appear gradually.

### Fine Tuning

6. You might pick up some speed by doing these things:
  - a. have fewer cases in the original collection, but not too few! (I have had good results using 250.)
  - b. make the seed be **random()**, and delete the **seed** slider
  - c. delete the first two graphs and the table (if you do that, save the file first, **Save As...** under a different name, and then delete.)
7. Your bifurcation graph would probably look better with smaller dots. Double click the graph's axes and choose a **Point size** in the graph's inspector. (You might try 3, or 1.)
8. To zoom into an area of the bifurcation graph, replace all the measures (in the **Collect Measures** tab of the **Measures from...** inspector), then double-click the  $r$  slider and choose the domain you want to study by entering a **Lower** and **Upper** value, and finally double-click the graph and make its x-axis match those.

## Project: Orbit Analysis of a “Dome” Function

Do a full orbit analysis of another “dome” function in the unit interval. Your function must be different from all other students’. Use the work we did on the logistic equation as a guide.

Possible functions:

Square root variations of the logistic function:

$$y = r\sqrt{x(1-x)}$$

$$y = r \cdot x\sqrt{1-x}$$

$$y = r(1-x)\sqrt{x}$$

Ellipse:  $y = r\sqrt{.25 - (x - .5)^2}$

Sine:  $y = r \sin(\pi x)$

Cubic:

$$y = r \cdot x(1-x)(p-x), \text{ with a specific } p \text{ somewhere between 1 and 4}$$

$$y = r \cdot x^2 \cdot (p-x)$$

Fourth degree:

$$r \cdot x(1-x)(p-x)(q-x), \text{ with a specific } p \text{ and } q \text{ somewhere between 1 and 4}$$

$$y = r \cdot x^2 \cdot (1-x)(p-x)$$

Variations of the cubic and fourth degree equations including square roots

0. Find the maximum value of  $r$  that keeps your function inside the unit square for values of  $x$  between 0 and 1.
1. For what value of  $r$  do we get extinction? equilibrium? periodic orbits? chaos? (for each of these behaviors give a description and examples)
2. Where are the fixed points? How do they show up graphically? Do they attract or repel? How is slope involved?
3. What are the points of bifurcation (at least from 1 to 2, from 2 to 4)? What does that mean? Explain this in terms of the graph of the function  $f(f(x))$ , etc.
4. Do you see any other patterns? Are there ways that your function differs from the logistic equation?
5. Try to find values of  $r$  that give orbits for which the period is not a power of 2, e.g. 3 or 6.
6. Include a short introduction (or conclusion) based on the readings, about how this sort of analysis fits in the recent history of science. However, make sure the math is the main part of your paper.

Use language correctly!

Your report should include answers to the questions, but do not write it as a series of answers. Think of it as an *illustrated article* to bring this information to someone who has taken Math 3, but not Infinity. (You can make illustrations in Fathom.)

You will be assessed based on how complete your analysis is, and how well you adhere to these guidelines.

## Iterating a Function of Complex Numbers

Reminder: the absolute value of a complex number is its distance from the origin.

0. What is  $|2+3i|$ ?  $|a+bi|$ ?

When iterating  $f(z)=z^2+c$ , the orbits will go towards infinity when  $|z|$  or  $|c|$  are greater than 2.

With smaller values of  $c$  and  $z$ , there are some interesting behaviors. Experiment with the following examples, starting *with seeds close to the origin*. Try many seeds.

1. For example,

$$c = .28 - .36i$$

$$c = .2 + .3i$$

$$c = -.5 + .5i$$

can give orbits that are attracted to a fixed point.

2. There is an interesting effect with round orbits circling around a central (fixed?) point for  $c = .35 - .35i$ .

3.  $c = .3 + .5i$  can yield a 4-cycle.

4. It appears that  $c=1+i$  gives only orbits that go to infinity. Can you find seeds that do not?

5. What happens when  $c = .4 + .2i$ ?

6. The easiest case is  $c = 0$ . Do an orbit analysis: for what seeds does the orbit go to infinity? For what seeds does it not? What is the boundary between those two sets? When the orbit does not go to infinity, what happens? (Hint: try  $0.6+0.8i$  as a seed.)

## Julia Sets

The next thing we are going to investigate is the following question: *for a given value of  $c$ , what seeds lead to orbits that escape to infinity? What seeds don't? What is the border between these two sets?* The border between the seeds that escape and those that do not is called the Julia set for this value of  $c$ , after the French mathematician Gaston Julia who investigated these before the invention of the computer.

Use software to find the Julia sets for the following values of  $c$ . Look for cases where the set consists of one connected piece, vs. cases where it breaks up into several islands. Look for fractal structures. Keep notes of interesting zooms and values of  $c$ .

1.
  - a.  $-1$
  - b.  $.3 - .4i$
  - c.  $.36 + .1i$
  - d.  $-.1 + .8i$
  
2.
  - a.  $.4 + .3i$
  - b.  $.4 + .2i$
  - c.  $.4 + .1i$
  - d.  $.45 + .25i$  (and values close to this one)
  - e.  $-.8 + .2i$
  - f.  $1 + i$
  
3.
  - a. values of  $c$  ranging from  $.24$  to  $.30$
  - b. values of  $c$  ranging from  $-.73$  to  $-.77$
  
4.
  - a.  $-.48 - .53i$
  - b.  $-1.25$
  - c.  $-.39 - .59i$
  - d.  $-.11 + .67i$
  
5. Find other interesting values of  $c$ .

Julia sets can be investigated for any complex function, for example the logistic function  $f(z)=rz(1-z)$ .

