

Geometry of Function Diagrams

Geometry Reminders

Fact P: In a quadrilateral, if a pair of opposite sides is both equal and parallel, then the quadrilateral is a _____. (You can prove this with the help of congruent triangles.)
Opposite sides of a parallelogram are equal.

Fact T: If the corresponding angles determined by a transversal are equal, then the two lines are _____. If two lines are parallel, then corresponding angles are _____.

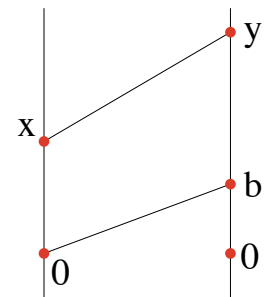
Fact AA: If two pairs of corresponding angles in two triangles are equal, then the triangles are _____.

Theorem: In the function diagram for $y = mx + b$, either all in-out lines are parallel, or they meet in one point

1. On the diagram of $y = mx + b$ shown on the right, how long are the vertical sides of the quadrilateral?

If $m=1$:

2. Show that the quadrilateral is a parallelogram (and therefore that the in-out lines are parallel.)



Since x is a generic input, you have proved that in the case where $m = 1$, *all* in-out lines are parallel to the one through 0 , and therefore to each other.

If $m \neq 1$, the quadrilateral is not a parallelogram, since opposite sides are unequal, and therefore the two in-out lines meet at a point we will call F .

Strategy for proof: We would like to prove that *all* the in-out lines go through that same point F , the focus. To do that, we will show that the position of F on the in-out line $(0, b)$ does not depend on the choice of x .

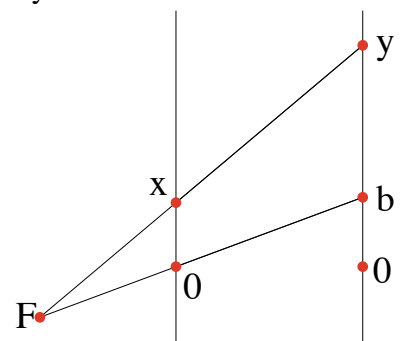
Consider the diagram below. Let us call the length $Ob = z$. This number does not depend on x . (It depends on b , and on how wide apart the axes are.)

4. Show that the triangles Fyb and $Fx0$ are similar, with ratio of similarity m .

It follows that $\frac{Fb}{F0} = m$, and therefore $\frac{F0 + Ob}{F0} = m$

5. Use algebra to show that $F0 = \frac{z}{m-1}$ if $m \neq 1$

So F is in the same position for any input x . In other words, all the in-out lines go through the focus, which is what we wanted to prove.



Theorem: If a function diagram has a focus, then the equation is of the form $y = mx + b$

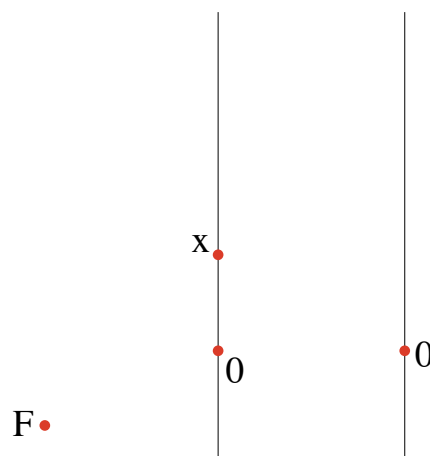
In other words, if all the in-out lines of a function diagram meet in one point F, then the image of all points x is given by the same formula, in the form $y=mx+b$.

Strategy for proof: we will use similar triangles to help us find a formula for the output corresponding to the input x.

1. Using a ruler, draw in-out lines for 0 and x. Label the output for 0 as b, and the output for x as y.

Call the ratio $Fb / F0 = m$.

2. Show that the two triangles are similar, with ratio m.
3. Find the ratio of the vertical sides, and solve for y.



Since m and b are constants that depend only on F's position, and since x was a completely generic point, we have proved the theorem.

Theorem: If all the in-out lines of a function diagram are parallel, then the equation is of the form $y = x + b$

In other words, if all the in-out lines are parallel, then the image of all points x is given by the same formula, in the form $y=x+b$.

Strategy for proof: we will use a property of parallelograms to help us find a formula for the output corresponding to the input x.

4. Why is the quadrilateral a parallelogram?
5. Use a property of parallelograms to show that $y = x + b$

Since b does not depend on x, and since x is a completely generic point, we have proved the theorem.

