LESSON 11.4

Irrational Numbers

In Lesson 2 you learned how to show that any terminating or repeating decimal can be converted to a fraction. In other words, you know how to show that terminating or repeating decimals are rational numbers.

If a decimal is neither repeating nor terminating, it represents an *irrational number* (one that is not rational).

For example, the number

$$0.010110111011110111110...,$$

created by inserting one, two, three, ... 1's between the 0's, never ends or repeats. Therefore it cannot be written as a fraction, because if it were, it would have to terminate or repeat.

1. Create an irrational number that is
   a. greater than 1 and less than 1.1;
   b. greater than 1.11 and less than 1.12.

While most numbers we deal with every day are rational, and even though there is an infinite number of rational numbers, mathematicians have proved that most real numbers are irrational.

$\sqrt{2}$ and $\sqrt{3}$ are familiar examples of irrational numbers. They cannot be written as a fraction having whole number numerators and denominators. In order to prove this, we will need to review prime factorization.

PRIME FACTORIZATION

Every whole number can be written as a product of prime factors.

Example: $990 = 99 \cdot 10$
$= 9 \cdot 11 \cdot 2 \cdot 5$
$= 2 \cdot 3 \cdot 3 \cdot 5 \cdot 11$

Note that 990 has a total of five prime factors. (Three is counted twice since it appears twice.)

2. Start the factorization of 990 by writing $990 = 3 \cdot 330$. Do you get the same prime factors?

3. Start the factorization of 990 a third way. Do you get the same prime factors?

Each whole number greater than 1 has *only one* prime factorization. Find it for the following numbers:

4. 12

5. 345

6. 6789

7. Find the prime factorization of several perfect squares. Try to find one having an odd number of prime factors.

Take the numbers 6 and 8. We have

$$6 = 2 \cdot 3 \text{ and } 8 = 2^3.$$

Six has two prime factors, an even number. Eight has three prime factors, an odd number. When we square them, we get:

$$6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$$
$$8^2 = (2^3)^2 = 2^6$$

8. Explain why any perfect square *must* have an even number of prime factors.

9. Explain why any number that is equal to twice a perfect square *must* have an odd number of prime factors.

THE SQUARE ROOT OF TWO

This section explains why $\sqrt{2}$ is not a rational number. The way we are going to do this is to show that if it were, it would lead to an impossible situation. This is called proof by contradiction.

If p and q were nonzero whole numbers and we had

$$\frac{p}{q} = \sqrt{2}$$

It would follow that

$$\left(\frac{p}{q}\right)^2 = \left(\sqrt{2}\right)^2$$

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

10. Explain each step in the previous calculations.

11. Explain why p^2 must have an even number of prime factors.

12. Explain why $2q^2$ must have an odd number of prime factors.

13. Explain why p^2 cannot equal $2q^2$.

We conclude that there can be no whole numbers p and q such that $\sqrt{2} = p/q$, and therefore $\sqrt{2}$ is irrational.

14. Use the same method to show that $\sqrt{3}$ is irrational.

15. Show why the method does not work to prove that $\sqrt{4}$ is irrational.

16. Does the decimal expansion of $\sqrt{2}$ terminate or repeat?

17. Does the line $y = \sqrt{2}x$ pass through any lattice points?

18. Do all lines through the origin eventually pass through a lattice point? Discuss.

19. Research π is probably the world's most famous irrational number. Find out about its history.

DISCOVERY *SUM FRACTIONS*

20. Find two lowest-term fractions having different denominators whose sum is 8/9.

DISCOVERY *COMPARING COUPONS*

21. Which is a better deal, 15% off the purchase price, or \$1 off every \$5 spent? Make a graph that shows how much you save with each discount, for various purchases from \$1 to \$20. Write about your conclusions.