

Decimals and Fractions

WRITING FRACTIONS AS DECIMALS

- How do you convert a fraction to a decimal number? Give examples.

When converting fractions to decimals, sometimes you get a *terminating* decimal like 3.4125, and sometimes you get a *repeating* decimal, like 7.8191919.... This last number is often written 7.819.

Problems 2 and 3 are easier if you work with lowest-term fractions.

- Exploration** For what fractions do you get a repeating decimal? Does it depend on the numerator or the denominator? (Hint: Pay attention to the prime factorization of the numerator and the denominator.)

- Exploration** For repeating decimals, is there a pattern to the number of digits in the repeating part? What is the longest possible repeating string for a given denominator? (Hint: Use long division rather than a calculator to explore this.)

- 💡 Explain why the decimals obtained as a result of a division *must* repeat or terminate.
- 🔑 Explain why some calculators give a decimal that does not seem to repeat for $2/3$: 0.6666666667.

WRITING DECIMALS AS FRACTIONS

Example: 3.4125 can be converted to a fraction by multiplying it by 10^4 , which gets rid of the decimal, and then dividing by 10^4 , which gets us back to the original number.

$$\frac{34,125}{10,000}$$

- Convert these decimals to fractions.
 - 6.0
 - 3.2
 - 0.015
 - 3.41

The case of repeating decimals is more difficult. Take $7.8\overline{19}$. Clearly, it is greater than 7.81 and less than 7.82. So it is between $781/100$ and $782/100$.

To find a single fraction it is equal to, we can rewrite it as:

$$\begin{aligned} &7.8\overline{19} \\ &= 7.8 + 0.0\overline{19} \\ &= 7.8 + 0.019 + 0.00019 + 0.0000019 + \dots \end{aligned}$$

Observe that:

$$\begin{aligned} 0.00019 &= 0.019(0.01) \\ 0.0000019 &= 0.019(0.01)^2 \end{aligned}$$

- Write the next term in the sum as a decimal, and as a product of 0.019 and a power of 0.01.

As you see, $7.8\overline{19}$ is the sum of 7.8 and a geometric sequence with first term 0.019 and common ratio 0.01. The sum of the first three terms of the geometric sequence can be written:

$$S = 0.019 + 0.019(0.01) + 0.019(0.01)^2$$

Multiply both sides by 0.01:

$$S(0.01) = 0.019(0.01) + 0.019(0.01)^2 + 0.019(0.01)^3$$

Subtract:

$$S(1 - 0.01) = 0.019 - 0.019(0.01)^3$$

Solve:

$$S = \frac{0.019 - 0.019(0.01)^3}{0.99}$$

Multiplying numerator and denominator by 1000:


$$S = \frac{19 - 19(0.01)^3}{990}$$

$$\begin{aligned}
 7.8\overline{19} &= 7.8 + S \\
 &= 7.8 + \frac{19 - 19(0.01)^3}{990} \\
 &= \frac{7.8(990) + 19 - 19(0.01)^3}{990}
 \end{aligned}$$

So

$$\begin{aligned}
 &= \frac{7741 - 19(0.01)^3}{990} \\
 &= \frac{7741 - 0.000019}{990}
 \end{aligned}$$

The sum is very close to $7741/990$.

8. Use the multiply-subtract-solve technique to add:
- the first 4 terms;
 - the first 5 terms.
9.  The numerator differs from 7741 by $19(0.01)^n$ if we add up the first n terms. Explain.

If we use large values for n , we find that the sum can get as close to $7741/990$ as we want. (Even with fairly small values of n , the sum of the first n terms differs from $7741/990$ by a very small number.) Mathematicians say that the whole infinite sum *converges* to $7741/990$, and they agree that we can write an equality:

$$7.8\overline{19} = 7741/990.$$

10. Check that this equality is correct, by converting the fraction back to a decimal.

A quick way to find the fraction is to use the multiply-subtract-solve technique on the decimal itself:

$$\begin{aligned}
 R &= 7.8191919\dots \\
 0.01R &= 0.0781919\dots
 \end{aligned}$$

Subtract:

$$\begin{aligned}
 R - 0.01R &= 7.8191919\dots - 0.0781919\dots \\
 (1 - 0.01)R &= 7.819 - 0.078
 \end{aligned}$$

(Notice that the infinite sequence of 19s disappeared.)

$$0.99R = 7.741$$

$$R = \frac{7.741}{0.99} = \frac{7741}{990}$$

11. Convert to a fraction.


a. $0.\overline{65}$ b. $4.\overline{321}$

RATIONAL NUMBERS

Definition: A *rational number* is a number that can be written as a fraction having an integer numerator and denominator.

Examples: 7, 0.5, and $-0.66666\dots$ are rational numbers, because they can be written as $7/1$, $1/2$, and $-2/3$.

Show that the following numbers are rational.

12. a. 0.3
b. $0.3333\dots$
13. a. 0.142857
b. $0.\overline{142857}$
14. a. 0.0909090\dots
b. 0.9090909\dots
15. a. 0.1111111\dots
b. 0.2222222\dots
16.  Mathematicians believe that $0.99999\dots = 1$. Explain why.