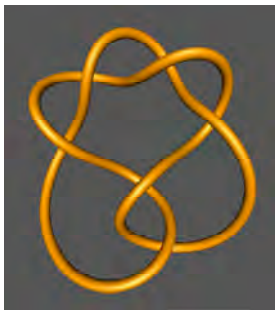
Knot Theory

Topology is a branch of mathematics that studies properties of figures that do not change when the figure is deformed continuously. Knot theory is the part of topology that studies figures such as the ones below. Chemists are interested in knotting long molecules to create substances with new properties. Biochemists are interested in knotting and unknotting DNA with the help of enzymes.



Definition: A knot is a continuous, closed loop in space, with no beginning and no end. This two-dimensional picture of a knot has six crossings.

1. Make sure you see all six crossings. Note that for this particular knot, they alternate: over, under, over, under, etc. Circle them on the picture.

2. For practice, use string, rope, or an extension cord to make this knot.

3. Make as many knots as possible that have a 2-D picture with *three or fewer crossings.* Sketch or photograph each one. If possible, do not undo any of the knots, so that you can see all your creations.

4. Among the knots you and your neighbors made, eliminate duplicates by getting rid of any that can be simplified. For example, if you have these two knots, keep the one with no crossings, because the other one can be simplified to it by just untwisting:

5. The knot with no crossings is called the *unknot.* It is one of only five knots that have *five or fewer crossings* when simplified. Try to find them all. Sketch them here:

6. Try to represent each of these knots by holding hands with other students. What is the smallest number of students needed to make each knot?

Teacher Notes

This activity is intended for a math club or an advanced high school class. Because it has no prerequisites, it may also work for a different audience.

The Worksheet

Materials: something to make the knots — string, rope, extension cords, etc. Optionally: sticks (toothpicks, etc.) and some way to connect them (tape, etc.)

Clarify the meaning of *crossing*, including that "pulling tight" doesn't change it. (A crossing is a feature of a two-D representation of a knot. Different representations will show different crossings. The *crossing number* is the smallest possible number of crossings for a given knot.)

Clarify *closed loop*. (A string with two ends is not a loop, and thus is not a knot, no matter how tangled. The "knot" in your shoelaces is not a knot in the mathematical sense. The combination of two or more knots is called a *link*, and is not a knot.)

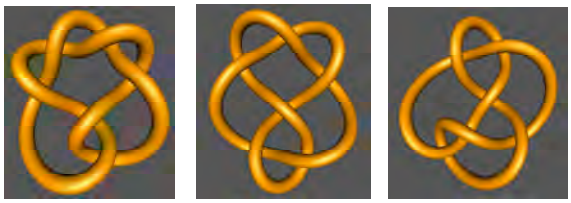
Clarify *equivalent knots*. (Knots that can be changed into each other without cutting / gluing.)

Reference: all the knots with five or fewer crossings. These are the knots that should show up in the answer to #5, possibly in different projections.

Unknot Trefoil Figure 8 5-star (no name)

Additional Problems to do before, during, after, or instead of the ones on the worksheet:

The 6's:



A: Make a bigger student-knot (one of the 6's, say) and see how many students can be removed without changing the knot.

B: After making it with a partner, describe how to make a knot (one of the 6's, say) "over the phone" to another student.

C: Try to represent each knot by using sticks. What is the smallest number of sticks needed to make each knot? This number is called the *stick number* of the knot. (There is a stick number table in Cromwell's *Knots and Links.*)

Resource: *The Knot Book* by Colin Adams (2001) is a comprehensive introduction to the subject.

[Knot pictures for this worksheet were made with the help of the program KnotPlot.]