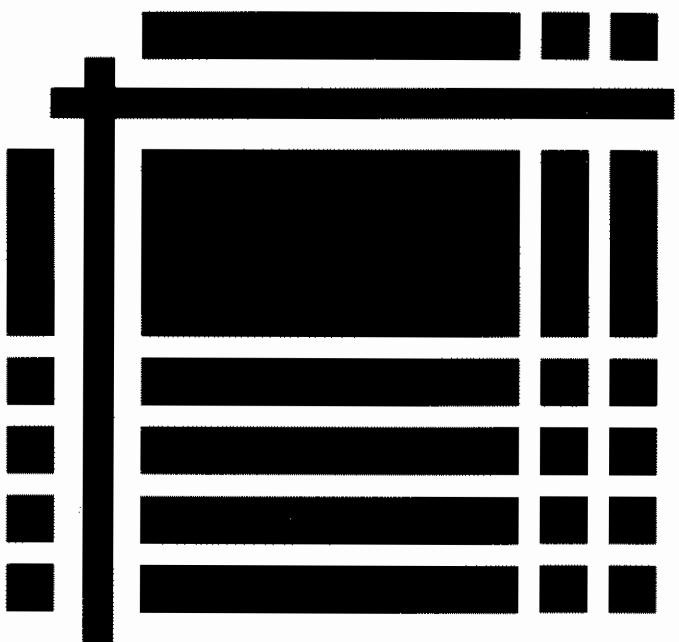
THE ALGEBRA LAB™

High School

A Comprehensive Manipulative Program for Algebra I



Henri Picciotto | www.MathEducation.page

THE ALGEBRA LAB™: High School

This book is dedicated to Mary Laycock, one of the great teachers of our time, who has done so much to promote the concrete approach to the learning of mathematics.

Cover design by JoAnne Hammer
Edited by Linda Charles

Acknowledgements

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Very special thanks to Urban School teacher Anita Wah. She provided not only inspiration, vision, and encouragement, but also many ideas, insights, and suggestions.

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The Algebra Lab™

The Algebra Lab™ is a complete manipulative program for teaching algebra concepts. It combines a unique algebra manipulative, The Lab Gear™, with two 144-page activity binders. The first binder, *The Algebra Lab™: Exploring Algebra Concepts with Manipulatives*, is intended for middle school students. The second binder, *The Algebra Lab™: A Comprehensive Manipulative Program for Algebra I* is intended for high school students. Either binder, combined with the Lab Gear, provides a powerful program for introducing, building, and extending the major topics of algebra.

The Lab Gear is an exciting manipulative designed to model algebra concepts. The set consists of ten types of blocks—three constant blocks (1-blocks, 5-blocks, and 25-blocks), and seven variable blocks (x-blocks, y-blocks, $5x$-blocks, $5y$-blocks, x^2-blocks, y^2-blocks, and xy-blocks). There is also a corner piece that helps students organize multiplication and division problems into rectangular arrays. The set of blocks comes in a specially-organized tub with enough blocks for three to six students.

There is a companion set of Algebra Lab Gear for the Overhead Projector. This transparent set of blocks allows the teacher or student to demonstrate examples, solutions, or explorations on the overhead projector. A unique classroom technique utilizes two projectors—the teacher (or student) at one projector, a student at the other. The teacher manipulates the blocks, while the student records with algebra notation, or vice versa. A great concrete-abstract bridge!

All blocks and activity binders are available from Creative Publications. Please see a current catalog for prices and ordering information.

The Philosophy

Algebra is the gate into high school mathematics and science classes, which in turn, are required for college admission. Few courses play as important a role in determining the future options of our students. And yet, many students never get the opportunity to take algebra, and among those who do, there is a high percentage of Ds and Fs. The Algebra Lab is an attempt to help change this grim situation. By offering a concrete, informal preparation for later symbolic and formal work, these manipulatives make algebra accessible to a broader constituency. To the stronger student, the Lab Gear offers an opportunity to gain a deeper conceptual understanding and to go beyond rote memorization of algorithms.

In grades 5-8, the Algebra Lab helps define those parts of the algebra curriculum that can be introduced as a sophisticated pre-algebra program for all students. In a first year algebra course, in 8th or 9th grade, the Algebra Lab helps introduce, illustrate, or review many concepts. It can also be used in remedial secondary classes at all levels.

The work is not easy, and you should not expect your students to pick it up spontaneously. Be sure to try the lessons on your own before trying to teach them. And if a few students are resistant to this unusual approach, be persistent—the blocks will probably grow on them.

The Algebra Lab will help you teach in three main concept areas:

- integer arithmetic
- the distributive law and factoring
- solving various types of equations

While the program does not attempt to cover the whole of algebra, these three areas cover a large part of any first-year algebra curriculum. Once they are mastered, the rest of the course is within reach.

The Lab Gear™

Because algebra is an extension of arithmetic, the Lab Gear has been designed as an extension of the most successful and effective of the manipulatives that are used to teach arithmetic—Base Ten Blocks. The Lab Gear is completely compatible with Base Ten Blocks—in fact the two can be used in conjunction with each other to teach algebra, arithmetic, or both. The use of 5-blocks and 25-blocks makes the Lab Gear more convenient to represent the generally smaller numbers that are encountered in algebra. The use of special blocks to represent two variables allows the Lab Gear to be used at a higher level of abstraction. The sizes of the x-blocks and y-blocks are carefully chosen to prevent the confusion that arises when a block (such as the Base Ten "rod") of whole number dimensions is used to represent a variable.

There are three other innovations of the Algebra Lab:

- a powerful combination of two methods to represent the minus sign (the minus area, and "upstairs")
- the workmat, for equation solving and algebraic fractions
- the corner piece, to help organize the rectangle model of multiplication and division

All of these components work together to create a unified concrete environment in which to learn the concepts of algebra.

Classroom Organization

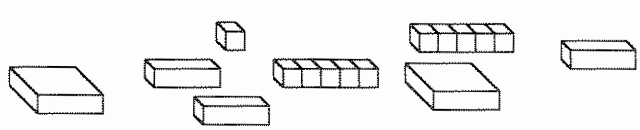
Depending on how many sets of blocks you have, organize your students in groups of three to six per tub. Within the groups, the students can work on one setup (e.g. a workmat or corner piece) individually or in pairs. In each tub, there are enough blocks for the simultaneous working out of three copies of most problems in the binder.

You probably have your own way of selecting groups. Try random groupings that change every two weeks. Random groupings are sometimes homogeneous, and sometimes heterogeneous, which allows you to take advantage of both types of arrangements. In a heterogeneous group, the stronger student can take the lead, or serve as a resource. In a homogeneous group of stronger students, there is the excitement of being able to do really fancy work fast. In a homogeneous group of weaker students, there is the comfort of not "feeling stupid" and not having to compare oneself to some star student. As long as the groupings are temporary, the students do not feel trapped, even if they do not like some of the students in the group.

Lab Etiquette

It is, of course, important that students respect the materials. A little time must be allotted at the end of each period for putting the Lab Gear back in the tub in an organized way so the next group will have no trouble finding the pieces they need. (Some group pride can be developed in relation to this.) Within their pairs, students should take turns working the problems. Beware of situations where one student does all the thinking and the other merely comes along for the ride. Pairs should periodically compare their answers and processes, just to make sure nobody is completely off-track. If differences emerge, they should try to resolve them by discussion. As a last resort, and only as a last resort, they can appeal to you for help.

Your role is to monitor the groups, acting more like a coach than a lecturer. If a group seems to lose its focus, get them back on task. If a group is too quiet, start a discussion for them. If a student is being rude to others, intervene. If an exercise is stumping an entire group, it is probably difficult for the whole class. At such moments, interrupt group work, and demand full focus on the overhead screen. Lead a class-wide discussion of the exercise. Whenever someone is speaking to the whole class, it is important for all to give their undivided attention. "No plastic in your hands" is the watchword for those times. Students who want to explain something to the group can come up to the overhead projector and demonstrate, or can talk while you demonstrate.



Using the Algebra Lab™ Lessons

The lessons in this binder are sequenced in a preview/view/review spiral. Typically, an idea is first encountered in its simplest manifestation, in the problem-solving context of an exploration. Later, a more formal lesson extends the idea into a broader range of cases. Finally, the idea gets used again as a component of more complex problems.

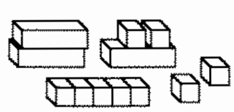
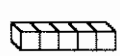
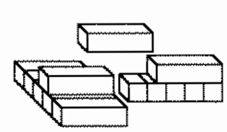
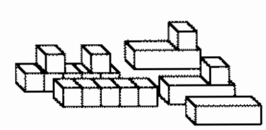
You are the best judge of which lessons are appropriate for your class. However, keep in mind that pedagogically, the most productive activities are the ones that require the students to think, experiment, and discuss, rather than merely reproduce a sequence of steps that have been demonstrated to them. A demonstration at the overhead is more effective if it follows the students' own attempts at solving a problem, rather than preceding them.

The answer is not the most important part of the problem. The process of finding it, and the concepts underlying the process are what this binder is about. Therefore, many answers are given in the body of the text so the students can check their work and verify that they are on the right track. (Or find out that they are not, and therefore need to start over or rethink their approach.) More answers are located in the back of the binder.

The classroom should not be quiet when students are using the Lab Gear. They will learn most if they are engaged in sharp discussions with each other. To keep their work organized, students should keep their own binder where pages with their work alternate with copies of the pages from this binder. Having previous pages to refer to can be invaluable to students when they are faced with a new variation on an old problem.

Students are often asked to explain their answers. This goes beyond merely "showing work," and can include writing sentences, or paragraphs, or drawing sketches. This is not easy for most students, and your coaching and coaxing is particularly important. It is worth the effort, because what they can explain they surely understand. To make sure students take this seriously, you should be sure to read what they write as often as possible. Note that explanations are not generally given in the solutions since answers will vary widely. Finally, make sure your homework assignments, quizzes, and tests include references to the work with the Lab Gear, so that the students see it as an integral part of the course.

The number of exercises in this binder should be adequate for your needs. If, however, you find that the number of exercises offered in one or another of the sections is insufficient, you can always create more yourself, or just use any algebra text as a source. But remember, it is usually preferable for the students to have a good in-depth discussion rather than to rush to do more exercises. Note that not all algebra exercises can be done with the blocks. That is also true of the exercises in this binder—not all of them can be done with the blocks. It is important for the students to remember that the Lab Gear provides a model of algebraic manipulation, but cannot encompass the flexibility of the symbolic notation that has evolved over centuries precisely because physical models proved insufficient. You will note that in many cases, the students will have to abandon the blocks in the middle of an algorithm, because, for example, there is no block to represent x/2. Conversely, you may be in a situation where the students, while working in a textbook, request access to the Lab Gear. If at all possible, you should allow them to use the blocks until they no longer need them.

Using the Explorations

In a traditional algebra class, the teacher often demonstrates how to carry out a specific algorithm, then the students are given a number of examples to practice the new skill. The problem with that approach is that often the students learn the procedures mechanically, the only justification for them being the teacher's authority, then quickly forget them.

It is in the Explorations that the Algebra Lab departs the most from this traditional approach. The Explorations include a minimum amount of instruction, relying instead on students' solving problems and constructing their own understandings of the logic that underlies the rules of algebra. The Explorations are usually quite challenging. They are designed to make such student-centered discovery possible, and are strategically placed in the binder to serve as a preview to the more formal approach used in the Lessons. If you are using the Algebra Lab in conjunction with a traditional algebra textbook, and don't have time to do everything in this binder, it is probably best to use mostly the Explorations.

The following are specific notes on the various types of Explorations:

- **Positive or Negative?** (pages 9, 23, 56, 75)
  The activities on these pages provide a way to practice simplifying expressions, and to reinforce some critical ideas about the sign of squares, cubes, and other expressions involving variables. The problems are chosen to help students think about (and hopefully avoid) some common mistakes, and clarify what happens with the sign of squares and cubes.

- **Minus Puzzles** (page 14)
  This exploration should force among the students a discussion of the correct use of the minus sign, and its representation with the Lab Gear. A good follow-up would be to ask the students to make up additional problems of this type.

- **Always, Sometimes, or Never True?** (pages 43, 53, 70, 75)
  True or False? (page 98)
  These are problems that get to the essence of the concept of variable. They are created to review or preview basic but tricky algebraic concepts and techniques, such as how one removes parentheses in different situations, or the Zero Product Principle.

  Some of the "Always, Sometimes, or Never?" problems are likely to be too difficult for your students to resolve when they are first encountered. The best approach with problems which cannot be classified on the spot is to save them until the time where the students have learned enough algebra to answer the questions themselves. It is healthy to have to deal with difficult problems, and it is important to develop a long term approach to solving them. Resist the temptation to give away the answer, though you can help the students think about it by suggesting various values for the variable.

  Student-created "Always, Sometimes, or Never?" problems (page 75) are likely to be even more difficult than the ones presented in chapters 3-5. Again, save the ones that cannot be categorized by the class until the time where they are equipped to deal with them.

- **Make a Rectangle** (pages 33, 55, 72, 95)
  Make a Square (pages 55, 95)
  These are factoring problems. They are more interesting than routine multiplications in the corner piece, because they involve more thinking, trial and error, and problem solving. In addition to factoring, they help introduce and develop the distributive rule, the main identities, and completing the square.

  Most of these activities avoid minus signs, which are reserved for the Lessons in Chapters 5 and beyond. One exception is the "Make a Rectangle" exploration on page 72, which is based on the difference of squares.

  The last activities in this series, on page 95, do not present a complete list of the blocks that are to be used. This deliberate ambiguity offers the students a final chance to pull together what they have learned in these explorations, right before the formal presentation of factoring.

- **Which is Greater?**
(pages 39, 52, 69, 81, 84)
These problems present a way to prepare students for the solving of equations and inequalities. When solving them, you may find that students want to delve deeper into the problems for which "it is impossible to tell." They may make statements like: "If x is greater than 2, then the left side is greater." Encourage them to discuss and write down these insights. They are essentially trying to solve an inequality. Questions like: "For what values of x would the two sides be equal?" lead into thinking about solving equations.

  A formal introduction to equation-solving is left until late in the book (Chapter 7). This is deliberate. Introducing this topic before students have a good feel for the meaning of algebraic symbols can lead to rote memorization of techniques with no sense of what they mean.

  When confronting the problems on page 52, your students will be tempted to do them without the blocks. If so, they will need to be extremely careful.

  On pages 69 and 81, the problems involve the use of the $\geq$ and $\leq$ symbols, which your students may not be familiar with. The first problems in these explorations are a check on the students' understanding of the effect of squaring on different numbers.

- **Area and Perimeter**
(pages 27, 38, 48, 69, 82, 96, 109, 122)
Volume and Surface Area
(pages 28, 53, 70, 82, 96, 111, 122)
The activities on these pages provide a connection with geometry, and a chance to practice combining like terms.

  To clarify the concept of surface area, the students should make paper jackets for the Lab Gear blocks, as in the example on page 28.

  If your students are getting frustrated when trying to find the perimeters of complicated Lab Gear figures, such as those on page 69, a method you could hint at is to find the perimeter of each component block. Then discuss whether the perimeter of the whole figure can be obtained by just adding the perimeters of the pieces. Of course, it cannot. Students can be guided to discover what needs to be subtracted from the sum to get the actual perimeter. The same method can be adapted to surface area calculations.

  Finding a way to continue the surface area sequences on page 70 will cause debate among your students. Accept any answer that acknowledges the repeated doubling of the volume, and the fact that the buildings are rectangular prisms.

  The perimeter and surface area sequences on pages 96, 109, and 111 turn out to be linear functions. You could have your students graph the resulting number pairs, with the position on the list as the x coordinate, and the perimeter (or surface area) as the y coordinate. The resulting graph will be a line, and can be used to predict perimeter (or surface area) for larger numbers of blocks. Another possible extension of this lesson is the introduction of subscript notation for sequences and discussion of the nth element of the sequence. Yet another extension is to try to analyze the relationship between the nature of the visual pattern and the parameters in the general formula that gives the value of the nth element. I owe the idea for this rich activity to Linda Dritsas of the Fresno, California, Education Center.

- **How Many Solutions?** (page 88)
For students to keep a sense of perspective, linear equations are introduced in the context of the questions "Always, Sometimes, or Never True?" and "How Many Solutions?" Too many students live under the erroneous belief that solving an equation always leads to the single inevitable dénouement: "x = some number."

- **Inequalities** (page 103)
This exploration should cause some vigorous discussion. Exercises 1-8 can be solved the same way that the corresponding equations would be solved. However, problems 9 and 11 should alert students to the fact that this technique does not always work. Students will probably need help answering question 12.

- **Solving for y, Solving for x and y**
(page 113)
More Solving With Two Variables
(page 114)
These explorations offer an informal introduction to solving simultaneous equations, as well as a review of basic linear techniques. They are likely to provoke lively discussions among the students.

How The Algebra Lab™ Fits into the Curriculum

The National Council of Teachers of Mathematics, and other leaders in math education are recommending:

- that algebra be introduced in elementary school
- that visual and physical models be a central part of the program
- that a problem-solving approach be used throughout math instruction
- that students work cooperatively and learn to speak and write mathematics
- that more emphasis be put on understanding, and less on complicated manipulations

The Algebra Lab incorporates all these ideas.

The Algebra Lab™: Exploring Algebra Concepts with Manipulatives can be your algebra program in grades 5-7. It should be used as part of a broader program that includes geometry, probability and statistics, and so on. In 9th grade, *The Algebra Lab: A Comprehensive Manipulative Program for Algebra I* can serve as an introduction to a full algebra course or it can be used in conjunction with any algebra textbook. It can also be a component in an integrated college preparatory curriculum. If your 8th grade students are closer to a standard 9th grade algebra class, use the high school binder. If they are not yet ready for a full-fledged algebra course, use the middle school binder. Noncollege-preparatory high school classes, can use either binder, with the goal of helping to prepare students to join the college-preparatory sequence.

If you're using this binder as part of a full Algebra I course, you should realize that work with the Lab Gear will take time. However, the extra depth of understanding gained means that you will need less time to cover the material. In general, it is best to do as much work as possible with the blocks early on in the school year, in order to establish a firm intuitive foundation to build on. However, if you will be alternating work in the Algebra Lab with work in your textbook, you need not try to match the topic sequence in the two programs. This binder has a logic of its own, and following it will mean that you will encounter some subjects earlier, and other subjects later than in your text. That is not a problem. In fact it is an opportunity for extra preview or review.

By its very nature, the Algebra Lab program uses a geometric model. Some geometric concepts are embedded in the very fabric of the program, particularly the concept of area, which is probably the most fundamental. Other geometric questions that are addressed are perimeter, volume, and surface area. This integration of another part of mathematics into the algebra program is not only a welcome breath of fresh air, but it also makes for a better understanding of the algebra concepts themselves.

A full algebra course should include more than the work with the Lab Gear. In fact, there are several important topics that cannot be addressed well with the Lab Gear. Some examples are graphing, work with exponents, work with polynomials of degree greater than two, deeply nested parentheses, "ugly" numbers, applications of algebra (except to problems of perimeter, area, and volume), and so on. At my school, we have the students work with polyominoes and pentominoes, the computer language Logo, computer graphing software, calculators, and with good problems from many sources, in addition to the Algebra Lab and a standard algebra textbook.

Please direct comments or questions about The Algebra Lab to the author, in care of Creative Publications, 788 Palomar Ave., Sunnyvale, California, 94086.

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Notes To the Student

In arithmetic, you have learned to work with numbers. Algebra is the logical extension of arithmetic. In algebra, you learn to work with symbols. Algebra is the language of all of mathematics and science. It is difficult to learn, but it is the key to so many possibilities in your life that it is worth the effort. In fact, you will probably enjoy it.

Mathematicians often use visual models to understand difficult ideas. In recent years, math teachers have been catching on to this idea, and as a result, students are learning a lot more. The Algebra Lab will help you get started in algebra. By the time you are finished with the material in this binder, you will no longer need the blocks. You will be able to do algebra the old-fashioned way, with pencil and paper.

In the future, more and more algebra will be done by computer. But what good would it do you to have a computer ready to do the algebra for you if you don't understand what algebra is? It would be as useful as a calculator to someone who didn't know the meaning of numbers.

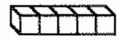
When working with this material, I would encourage you to always have the Lab Gear blocks within reach, and to follow all the examples given with your own blocks. It is very difficult to really learn math in silence. Be prepared to discuss with your classmates and teachers any questions or ideas you may have. Finally, have a pencil and paper available to record your calculations and the answers to the exercises. The better you organize your work, the better you will understand it.

Henri Picciotto, Author

The Algebra Lab Gear™

These are the blocks that make up a set of Algebra Lab Gear.

1



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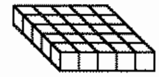

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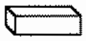
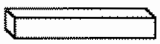


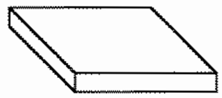
x

x^2

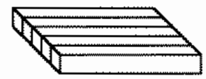
$5x$



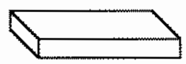
y



y^2



$5y$



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Chapter 1 Meeting the Algebra Lab Gear™

This chapter introduces all of the Lab Gear, with the exception of the corner piece.

New Words and Concepts

This chapter does not assume an in-depth understanding of signed number arithmetic, which is introduced in Chapter 2, but it does introduce several important concepts of algebra, which will be returned to throughout the book. These include:

variables and **constants**
combining like terms
the various meanings of the **minus** sign
squaring a quantity
cancelling

Teaching Tips

If your class has had experience with basic algebra, you may want to speed through this chapter. However, do not skip too many exercises or too much reading, since your students need to master the Lab Gear vocabulary, and the basic Lab Gear techniques, before going on. In particular, make sure they know the name of each of the blocks, and that they understand the two ways to illustrate the minus sign and its three meanings. In any case, it is recommended that you do not skip the Explorations. (For notes on the Explorations, see page ix.)

The corner piece will make its first appearance in Chapter 3. If your students are curious about its purpose, encourage them to guess.

Lesson Notes

- **Lesson 1,** The Blocks, page 3: Show the students how to use the x- and y-blocks to measure the sides of the rectangles in order to identify the x^2-block, the y^2-block, and the xy-block.
- **Lesson 2,** Sketching the Lab Gear, page 5: Discourage perfectionism here. The sketches should allow one to recognize the blocks, but they need not be works of art.
- **Lesson 3,** Variables, page 6: This is a crucial activity to establish the concept of variable. Note that no negative numbers are used, yet.
- **Lesson 4,** Like Terms, page 7: This lesson should help prevent mistakes of the type $x + x = x^2$. If students want to do this lesson by looking at the figures, without using actual blocks, let them, unless they still have trouble recognizing the blocks. At this stage, do not use abstract arguments such as the distributive rule to explain combining like terms—the blocks make it clear enough.
- **Lesson 5,** Minus, page 9: Misunderstandings about the meaning of the minus sign are a major obstacle to students' understanding of algebra. It is best to start talking about this confusing but powerful symbol explicitly early on. You may want to lead a class discussion of the ideas in this lesson.
- **Lesson 6,** Minus with the Lab Gear, page 10: Be sure to make copies of the workmat (page 131) for all your students. If they keep their work in a binder, make sure that the workmat has holes punched in it.

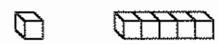
  The Lab Gear does not use color to differentiate positive from negative numbers, because that approach cannot be generalized to variables. ($-x$ is not necessarily negative.) The two methods presented in this lesson, used in combination, make it possible to have a unified manipulative environment for algebra. Using color, or using just one of the methods at a time puts enormous limits on the flexibility of an algebra manipulative program.
- **Lesson 7,** Opposites, page 12: Stress that the same number can be represented in many ways. Cancelling and adding zero ("uncancelling") both turn out to be critical skills in algebra and in manipulating the Lab Gear.
- **Lesson 8,** More on Minus, page 13: Notice that no attempt is made here to teach or review signed number arithmetic. Instead, students are shown how to simplify expressions that involve the minus sign by manipulation of the Lab Gear.

The Blocks

Look at your Lab Gear blocks. There are two kinds of blocks, yellow and blue.

The Yellow Blocks

The yellow blocks represent whole numbers, such as 1 or 5.

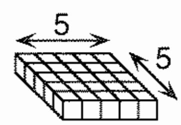


Find blocks in your Lab Gear that you can use to show these quantities.

1. 3
2. 13
3. 31
4. Write some numbers that *cannot* be represented by the Lab Gear. Discuss your answers with your classmates.

You will soon learn to use the Lab Gear for negative numbers. Later, you will use the Lab Gear to work with fractions.

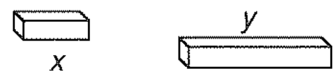
Notice that the block that represents 25 is a 5 by 5 square. In algebra, the multiplication 5 times 5 is written $5 \cdot 5 = 25$, or $5\,(5) = 25$. Do not use $\times$ to indicate multiplication—it could be confused with the letter *x*.



5. Use the blocks to show 30 as a rectangle. Find four different ways. For each way, write a multiplication equation.

The Blue Blocks

The blue blocks represent *variable quantities*. All of them are related to these two blocks.



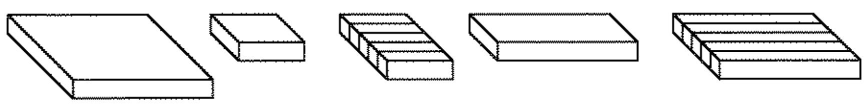
Variables are usually named by letters. Since the names x and y are used most often in algebra books, they have been chosen to name the variables in the Lab Gear.

6. Write a way to remember which block is x, and which is y.

In algebra, 5 times x is usually written $5x$. (You do not need to write the multiplication dot between a number and a variable, or between two variables.)

7. Find blocks that show these quantities.
   a. 5 times x (write $5x$)
   b. 5 times y
   c. x times y (write xy)
   d. x times x (write x^2)
   e. y times y

   Label the blocks in this figure.



The expression x^2 is read x **squared** or *x to the power 2.* It means *x multiplied by itself.* In algebra, x and y are called variables because they can stand for different numbers at different times.

Copy and complete these sentences.

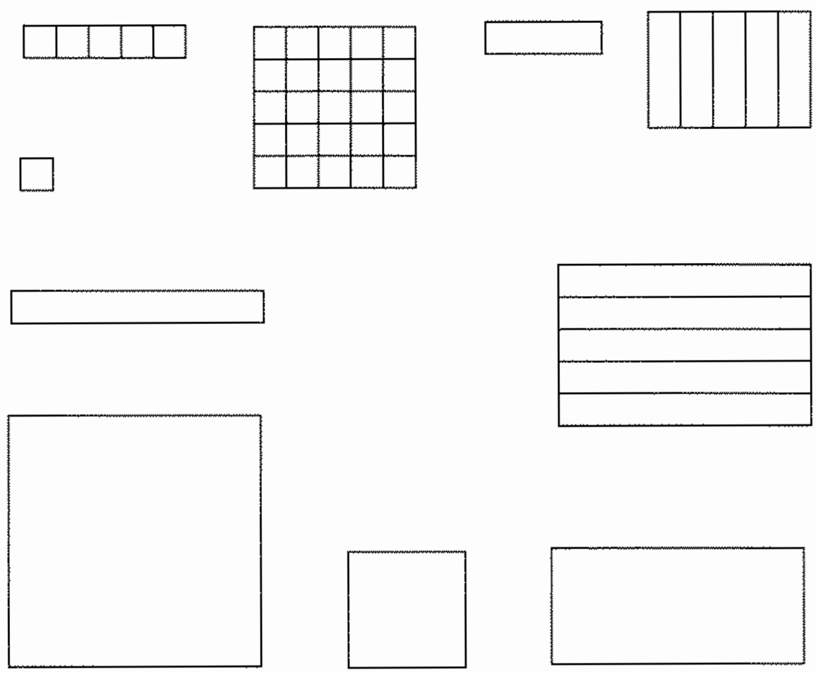
8. If $x = 6$, then $5x =$ ______, and $x^2 =$ ______.
9. But if $x = 6.2$, then $5x =$ ______, and $x^2 =$ ______.

We agree that the x-block and the y-block can represent any numbers, including negative numbers and fractions. Even though the x-block is shorter than the y-block, it could represent a greater number. Or x could be equal to y, or less than y.

10. Explain why x^2 is read *x squared.*

Sketching the Lab Gear

You will often be asked to sketch solutions to problems. It is easier to sketch the Lab Gear blocks two-dimensionally, as they appear when seen from above.

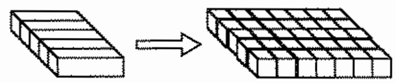


1. Label each of these block sketches.
2. Put a check next to the ones that are **constants** (not variables).
3. On separate paper, practice sketching each of the blocks.
4. Next to each of your sketches, write what the value of the block would be if $x = 3$ and $y = 2$.
5. Draw three-dimensional pictures of some of the Lab Gear blocks.

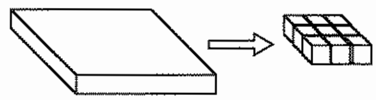
Variables

The blue blocks represent variables. They can stand for different numbers. When you know the value of the variable you can replace the blue blocks with the appropriate yellow blocks. Look at these examples.

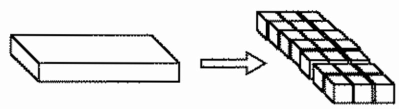
Here is $5x$ if $x = 7$.



Here is y^2 if $y = 3$.

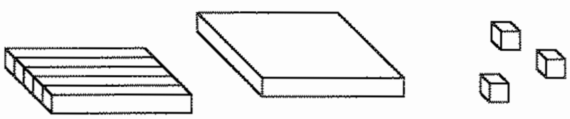


Here is xy if $x = 7$ and $y = 3$.



To answer these questions, first put out blocks to match each figure. Then, when possible, replace the variables with the given constants, and count what you have.

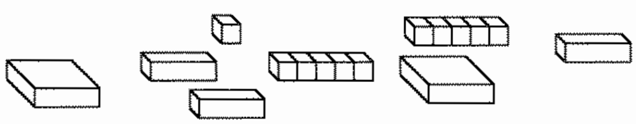
1.



Write what the blocks show if:

a. $y = 1$
b. $y = 2$
c. $y = 0$
d. $y = 1.3$

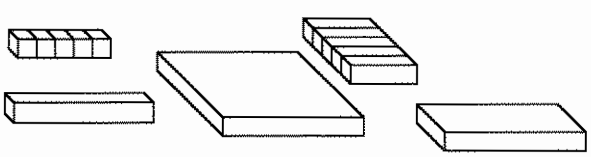
2.



Write what the blocks show if:

a. $x = 2$
b. $x = 1$
c. $x = \frac{1}{2}$
d. $x = 1.5$

3.



Write what the blocks show if:

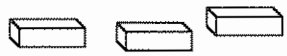
a. $x = 5$ and $y = 4$
b. $x = 3$ and $y = 1$
c. $x = 1.2$ and $y = 2$
d. $x = \frac{1}{2}$ and $y = 4$

4. Make up a problem of this type for another student.

Like Terms

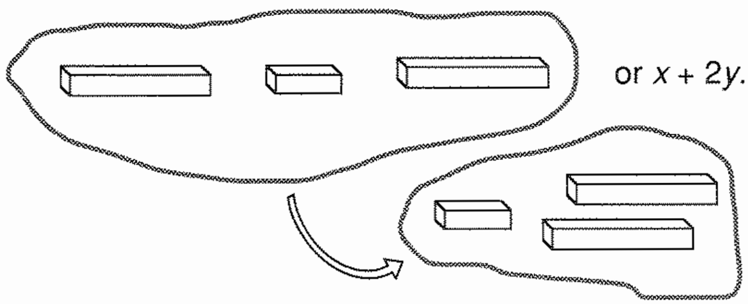
You can write an algebraic expression that names a group of blocks many ways. If you group the blocks together that have the same size and shape, that is a way to show **combining like terms**. Look at these examples.

This quantity is written $x + x + x$,



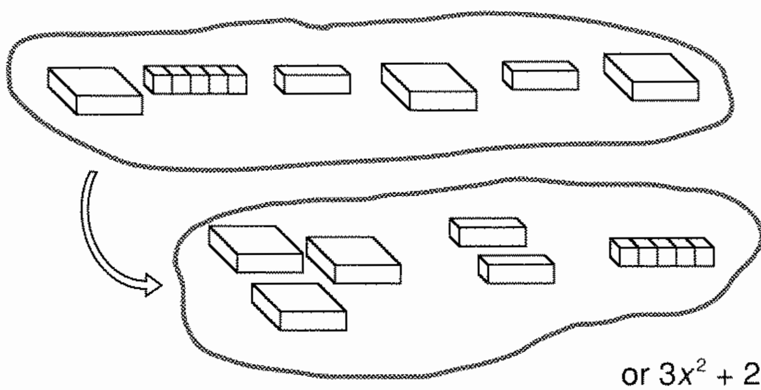
or $3x$.

This quantity is written $y + x + y$,



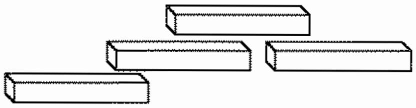
or $x + 2y$.

This quantity is written $x^2 + 5 + x + x^2 + x + x^2$,



or $3x^2 + 2x + 5$.

For each example, show the figure with your blocks, combine like terms, then write the quantity the short way.

1. 

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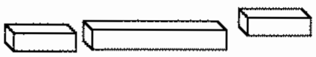
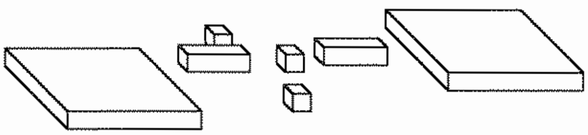
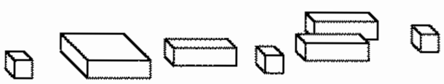
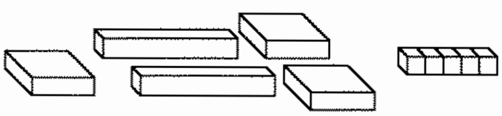
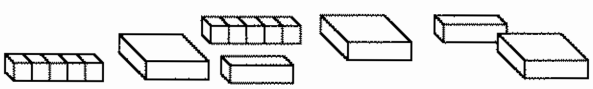
3. 

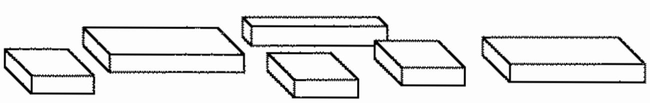
4. 

5. 

6. 

7. 

8. 

9. 

Minus

The **minus** sign can mean three different things, depending on the context.

- In front of a number, and only there, it means *negative*.
  For example, -2 can mean negative 2.
- In front of any expression, it means *the opposite of*.
  For example, -2 can mean the opposite of 2, which is negative 2.

  Also, $-(-2)$ means the opposite of -2, which is 2. (Notice that $-(-2)$ could be written $- -2$. The parentheses are added to make it easier to read.)

  For another example, $-x$ means the opposite of x. So, if x is positive, $-x$ is negative. If x is negative, $-x$ is positive.

  Also, $-(2x + 1)$ means the opposite of $2x + 1$. So, if $2x + 1$ is positive, $-(2x + 1)$ is negative. If $2x + 1$ is negative, $-(2x + 1)$ is positive.
- Between two expressions, it means *subtract the second expression from the first one.*

  For example, $x - 3$ means subtract 3 from x. Also, $3 - x$ means subtract x from 3.

For each expression, write an explanation of what the minus signs mean.

1. -3	4. $3 - 7$	7. $3 - x$	10. $-(4x + 1)$
2. $5 - 2$	5. $-(4 + \frac{1}{2})$	8. $-(-x)$	11. $y - 5$
3. $-(-5)$	6. $-(2 - 6)$	9. $2y - y$	12. $y - (-x)$

Exploration 1 Positive or Negative?

1. Write the value of $-x$ if:
   a. $x = 2$
   b. $x = -3$

2. True or False?
   a. $-x$ is always negative
   b. $-x$ can be positive
   Explain your answers.

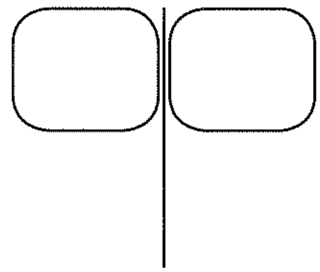
Minus with the Lab Gear

With the Lab Gear, we show the minus sign in two different ways.

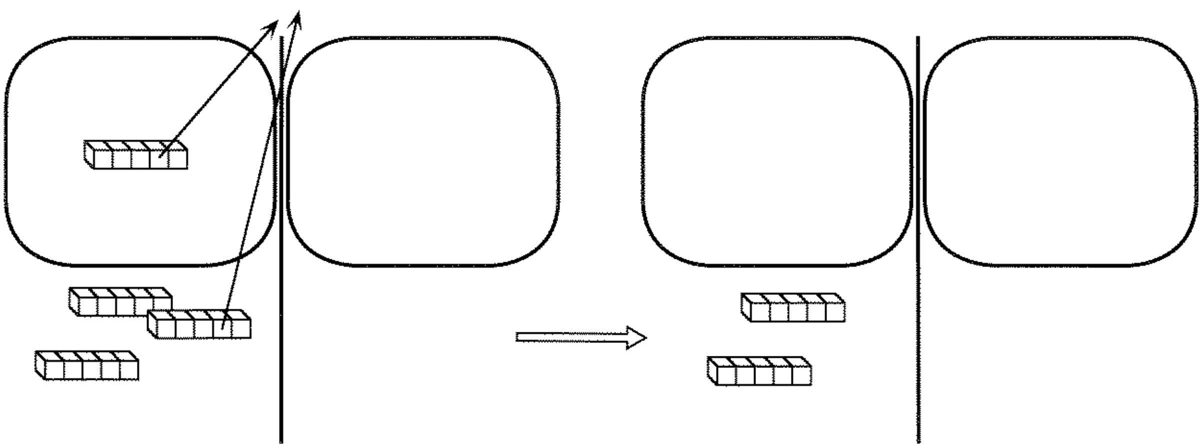
- We can put blocks inside the *minus area* of the workmat.
- We can put blocks *upstairs* (on top of other blocks).

The Minus Area

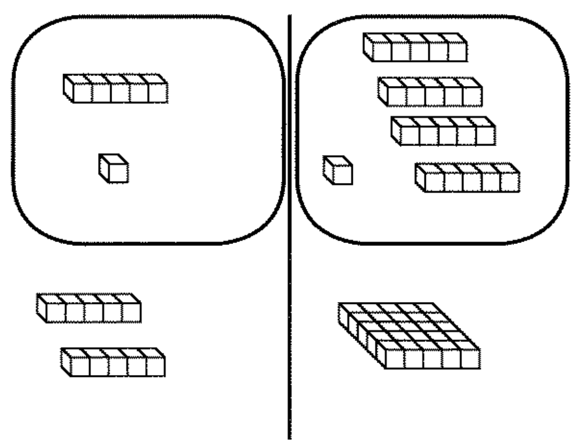
Look at your workmat. The rectangles with rounded corners represent the *minus areas*. If you remove matching blocks from inside and outside the minus area, the remaining blocks show the simplest way to write the expression.



For example, the subtraction equation 15 – 5 = 10 can be shown this way on the workmat.



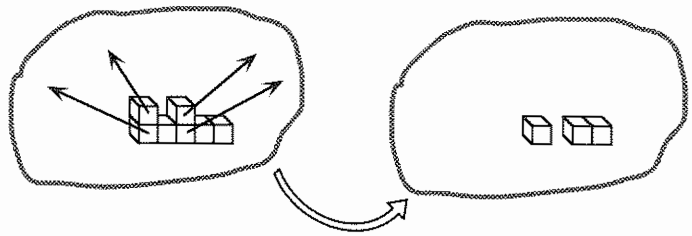
The problem above was worked on one side of the line on the workmat. When using both sides, the line usually represents an equal sign. For example, this workmat shows the equality 10 – 6 = 25 – 21.



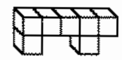
This is an equality since both sides, once simplified, represent the same number.

1. Mark the blocks that can be removed. Write what both sides show when simplified by removing blocks.

Upstairs



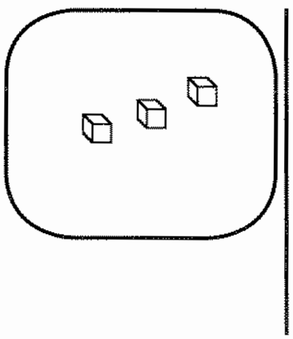
This figure shows 5 – 2. Notice that the *uncovered* part of the bottom block equals 3. If you remove matching upstairs and downstairs blocks, you will be left with three downstairs blocks. This is how we show 5 – 2 = 3 with upstairs and downstairs blocks.

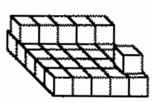


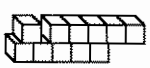
2 – 5 = –3

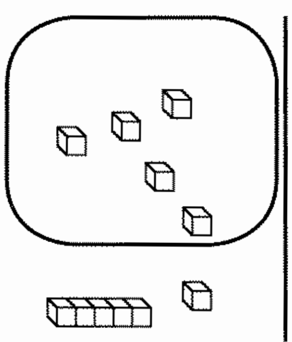
This figure shows 2 – 5. If you mentally remove matching blocks downstairs and upstairs, you are left with 3 upstairs blocks, or –3. We can only do this mentally, however, since blocks cannot float in mid-air.

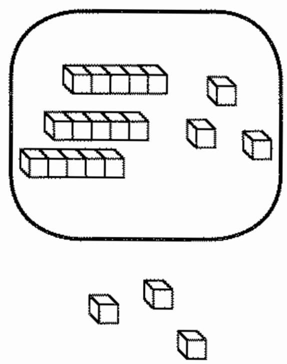
For each figure, write a subtraction expression.

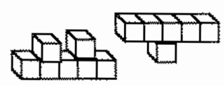
2. 

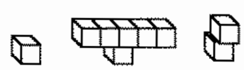
3. 

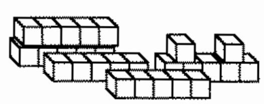
4. 

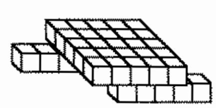
5. 

6. 

7. 

8. 

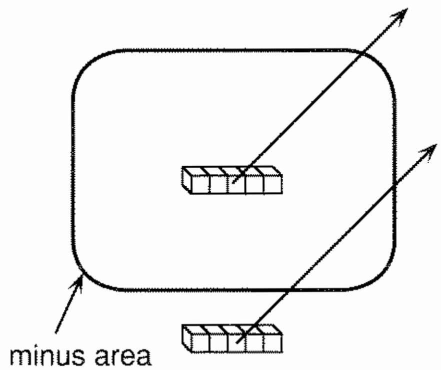
9. 

10. 

Opposites

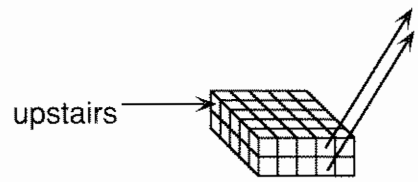
You will use the workmat to work with **integers**. Integers are positive and negative whole numbers, and zero. When you add opposites, such as 3 and –3, you get zero. When modeling algebraic expressions on the workmat, we can use informal language and say that 3 and –3 can be **cancelled**. Look at these two examples of cancelling.

This example of cancelling uses the minus area.



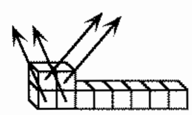
$5 + (-5) = 0$

This example of cancelling shows blocks *upstairs*.



$25 + (-25) = 0$

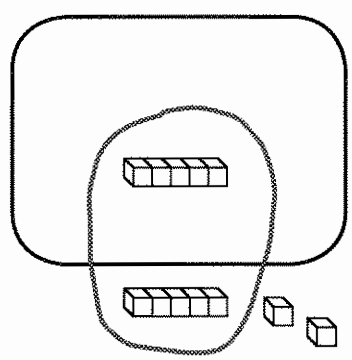
Usually, after cancelling, something is left. This example shows what happens if we start with 8 downstairs, and 2 upstairs. After cancelling, 6 blocks are left downstairs. We say that 8 downstairs and 2 upstairs is a way of showing the number 6.



1. a. What is left after cancelling?

   b. What number do 9 blocks in the minus area, and 2 blocks outside show?

Sometimes, it is useful to show a number with more blocks. For example, the number 2 can be shown with two one-blocks outside the minus area. But even after adding a five-block in the minus area and a five-block outside the figure still shows 2.

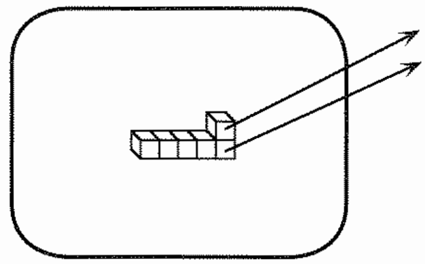


2. Sketch two other ways to show the number 2.

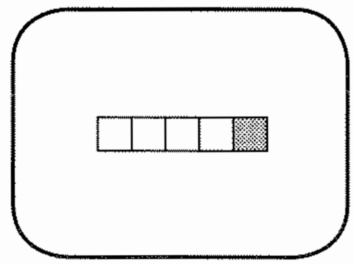
More on Minus

Minus signs can be a source of many errors for algebra students, but using the Lab Gear makes it easier to think about minus. Consider what happens when we have upstairs blocks inside the minus area.

For example, in this figure, we have the quantity (5 – 1) inside the minus area. This is written –(5 – 1). Since the upstairs and downstairs blocks cancel, the quantity simplifies to –4.



For a quick sketch of this, draw the 5-block as it is seen from above, and shade in the part that is covered by the 1-block.

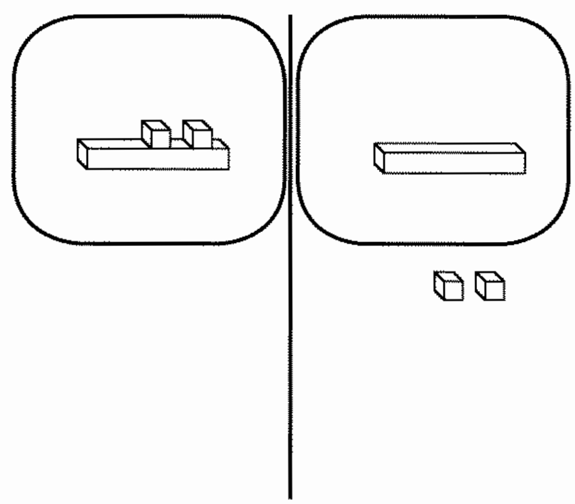


But this sketching method does not work if the upstairs block is bigger than the downstairs block. In that case, try for a three-dimensional picture.

1. Find two other ways to show –4 using only a 5-block and a 1-block. Sketch your solutions.

Do not stack Lab Gear blocks more than two levels high. Two levels is enough to illustrate many ideas of algebra, and will keep things clear. More would be confusing.

Now consider this figure. The left side shows the opposite of (y – 2), We write, –(y – 2). The right side shows the opposite of y, and 2. We write, $-y + 2$. The vertical line means =. So to record the figure, we can write, $-(y-2) = -y + 2$.



2. Do you think this statement is always true? Explain your answer. (Try different values for y, such as –5, –1, 0, and 2. See if they make the statement true.)

3. a. Sketch a figure to illustrate the statement: $-(5 - x) = -5 + x$
   b. Do you think this statement is always true? Try different values for x to test it, such as –7, –3, 0, 2, and 8.
4. Look at the figure for problem 2, and at the figure you drew for problem 3. What can you say about upstairs blocks in the minus area?

Exploration 2 Minus Puzzles

Use your Lab Gear to solve these puzzles.

1. Find four ways to show 3 with three blocks.
2. Find four ways to show –8 with four blocks.
3. Show –9 with three blocks.
4. Show –9 with five blocks.
5. Show –9 with seven blocks.

Chapter 2 Computing with Signed Numbers

This chapter is an introduction to integer arithmetic.

New Words and Concepts

The usual approach to work with signed numbers (teaching of rules, sometimes combined with number line techniques) is often frustrating, because so many students seem to forget the rules, even though they are given endless practice. On the other hand, the students who "get it," quickly get bored with the drills.

The Algebra Lab features a different approach. Instead of being taught signed number arithmetic, the students are presented with a model for the four operations. The model is easy to learn, because it is a natural extension of concrete models most students have already mastered for these operations. In fact, once learned, the model is almost impossible to forget, unlike the rules and number line techniques, which are confusing and easy to mangle. Moreover, the work with the blocks keeps more students motivated, at all levels of proficiency.

The commutative law is not mentioned by name, but students are asked to observe what happens when switching the order of the terms for each operation.

After working with the blocks for a while, you will find that many students discover the rules for themselves, or learn them from each other. Knowledge that is earned the hard way by one's own thinking and interaction with peers is much more securely anchored than techniques that have been learned by rote.

The geometric concepts of **perimeter**, **area**, **volume**, and **surface area** appear for the first time in this chapter.

Teaching Tips

In grades K-4, students should have been introduced to whole number, fraction, and decimal arithmetic with manipulatives such as Base Ten Blocks. Such background is certainly helpful in understanding the manipulative approach to the arithmetic of signed numbers.

If your students already know signed number arithmetic, there is little point in working through this chapter, except for Lesson 4 and the Explorations. An interesting approach with such students is to ask them to figure out a way to use the Lab Gear to explain the rules of signed number arithmetic. If they cannot figure it out, show them the models, but do not assign too many drill problems.

In general, students should be able to perform most arithmetic calculations in this program mentally. However, a student whose arithmetic is shaky should not be prevented from learning algebra. Using a calculator should allow such a student to temporarily get around this handicap. Of course, all students should also be allowed to use calculators if they need them on lengthy, multi-digit computations.

Do not make the mistake of skipping Explorations 2 and 3! The geometric content is an enhancement to an algebra course, and offers opportunities to apply algebraic ideas.

Lesson Notes

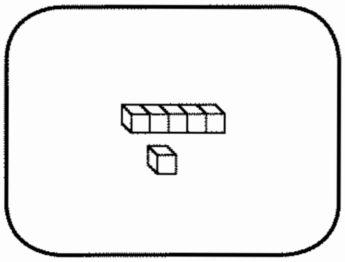
- **Lesson 1,** Addition, page 16: This lesson is based on the cancelling concept from Chapter 1, Lesson 7.
- **Lesson 2,** Subtraction, page 18: This lesson is based on the "uncancelling" concept from Chapter 1, Lesson 7.
- **Lesson 3,** Multiplication, page 21: The "rectangle" interpretation of multiplication is introduced here. This is a crucial conceptual tool, which will be developed further (in conjunction with the Lab Gear corner piece) starting in Chapter 3.
- **Lesson 4,** Squaring Numbers, page 23: Most students will not learn the concepts on this page by just reading it. It is important that there be some verbal discussion of these ideas.
- **Lesson 5,** Generalizing, page 24: Later in the year, if your students have trouble with signed number arithmetic, refrain from reminding them of the rules. This would only perpetuate their dependence on you. Instead, allow them to use the blocks to refresh their memories, or to look up what they wrote at the end of this chapter. In most cases, this will do the trick.
- **Lesson 6,** Division, page 25: The key to this lesson is a solid understanding of multiplication, and of the relationship between the two operations. See Chapters 3-7 for Lab Gear division techniques on algebraic expressions that involve variables.

Addition

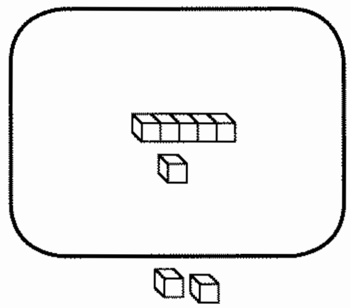
To model addition with the Lab Gear, *put on* the first number, *put on* the second number, *cancel* what you can, and *count.*

Look at this example, –6 + 2.

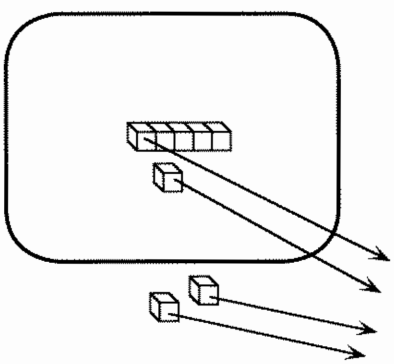
- *Put on* the first number. To show –6, put 6 in the minus area.



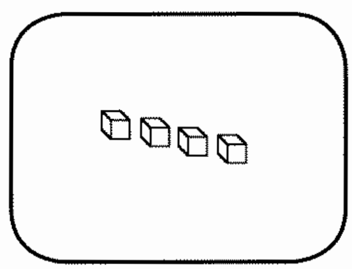
- *Put on* the second number. To show 2, put 2 outside the minus area.



- *Cancel* 2 and –2.



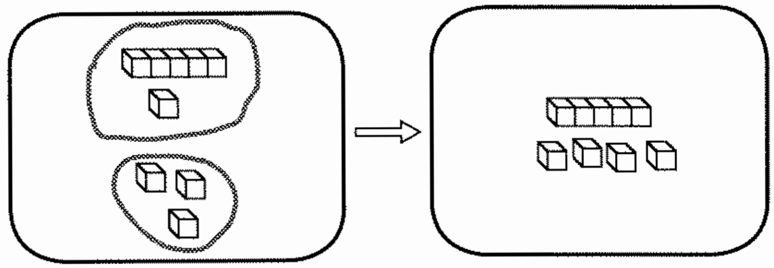
- *Count* to find the answer. –6 + 2 = –4



1. Does the order that you put on the numbers matter? Try it by modeling 2 + –6, and compare the result.

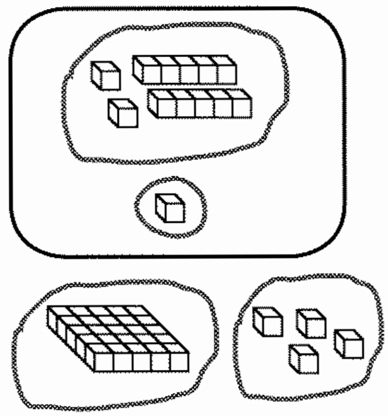
Sometimes, there is nothing to cancel.

2. Write the addition shown by this figure.



If you are adding several numbers, just put them all on the mat, cancel, and count.

3. Write the addition shown by this figure.



Use the Lab Gear to model these additions. Sketch the process in four steps (put on first number / put on second number / cancel / final solution). The sketches should include the minus area. You can sketch the blocks in two dimensions.

4. $-5 + 2$
5. $-15 + (-3)$
6. $-8 + 9$

Use the Lab Gear to find these sums.

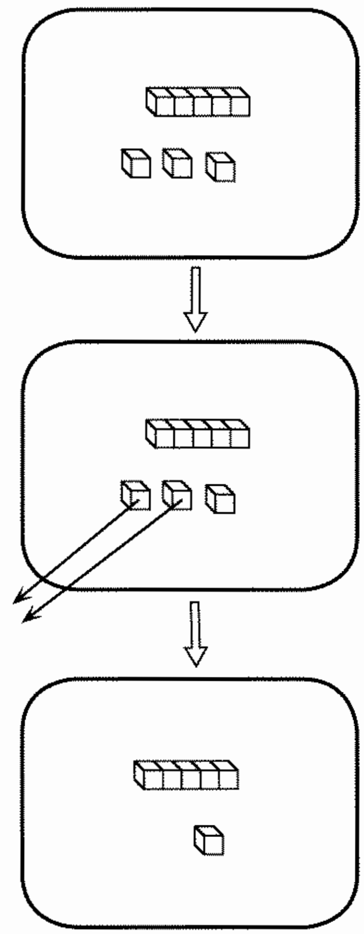
7. $10 + -4$
8. $-25 + 5$
9. $-12 + -6$
10. $10 + (-8) + 3$
11. $-25 + 11 + (-4)$
12. $-15 + 20 + (-7) + 2$

Subtraction

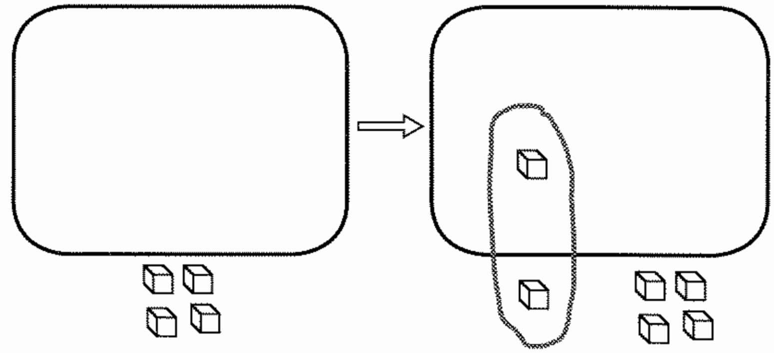
To model subtraction with the Lab Gear, *put on* the first number, *take off* the second number, and *count*.

This example shows that –8 – (–2) = –6.

- Put on –8.
- Take off –2.
- Count to find the answer. –6

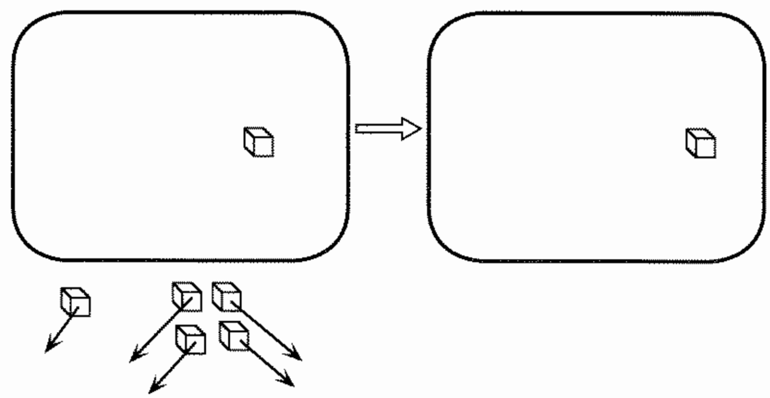


Sometimes, there are not enough blocks to take off. For example, 4 – 5. The solution is to show 4 differently by *adding zero.*



As long as you add the same amount inside the minus area and outside it, you have not changed the quantity on the workmat. If you cancelled the blocks you put on, you would be back to the original amount. (Adding zero could be called "uncancelling.")

Notice that now it is possible to take off 5, and you can see that the answer is –1.



Perhaps you knew what the answer was going to be, and you are asking yourself, why do so much work to get it? Keep in mind that the point of working with the Lab Gear is not just to get the answer, but to understand why it turns out the way it does. Work these easy problems carefully; you will soon be doing similar ones with variables. You will then see how working with variables is just an extension of the work you're doing now with integers.

Use the Lab Gear to model these problems.

1. $-7-(-3)$
2. $-3-(-7)$
3. In problem 2, the order of the numbers was reversed. How did that affect the answer?

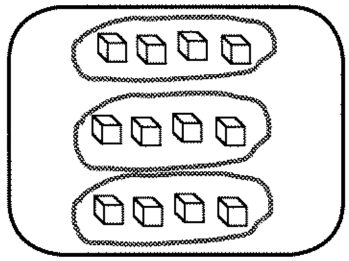
The adding zero trick should help you do some of these problems. For problems 4-7, sketch the process in four steps (put on / add zero (if you need to) / take off / final solution).

4. $-2-8$
5. $-4-(-2)$
6. $-8-3$
7. $7-(-6)$
8. $-5-9$
9. $11+(-6)$
10. $-1+10$
11. $20-25$
12. $13-3$
13. $-17+3$
14. $-2-21$
15. $-15-5$
16. $-5-(-15)$
17. $6-17$
18. $18-9$

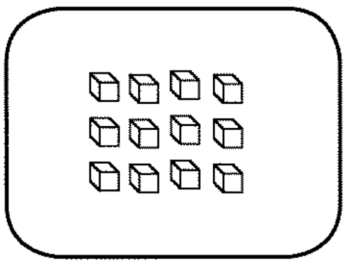
Multiplication

To model multiplication with the Lab Gear, *always start with an empty workmat.* For the first example, consider the multiplication 3 (–4).

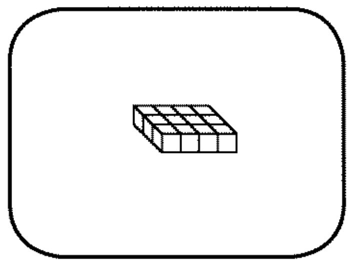
- Look at the first number. If the first number is positive, *put on* the second number the number of times indicated by the first number. Here you put on three sets of –4.



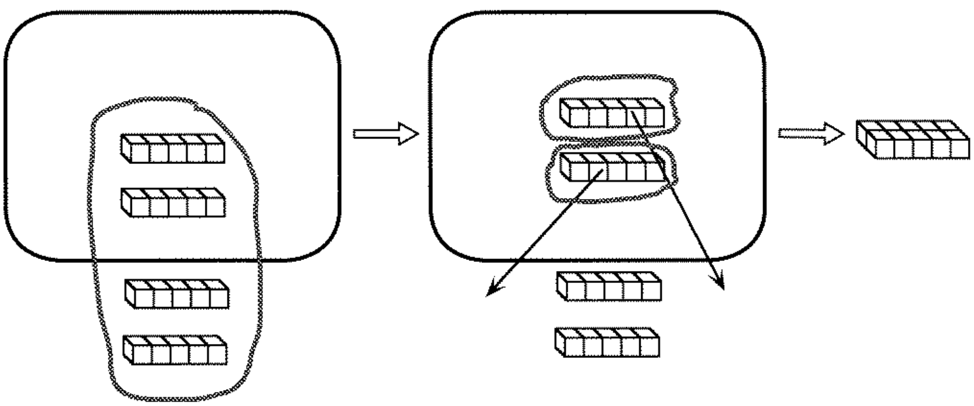
- *Count* to get the final result. –12



- *Make a rectangle* to show multiplication. 3 (–4) = –12



For the second example, consider the multiplication –2 (–5).



- If the first number is negative, *take off* the second number the number of times indicated by the first number. However, since you're starting with an empty mat, there is nothing to take off, so you must use the adding zero trick. In this case, add 10 and –10, then *take off* two sets of –5.
- *Count* to get the final result.
- *Make a rectangle.* –2 (–5) = 10.

Look at the examples on page 21. Use the Lab Gear in the same way to model problems 1 and 2. Notice that the order of the numbers has been reversed.

1. –4 (3)
2. –5 (–2)
3. Tell how changing the order affects the answer.

Notice that at the end of a multiplication problem, you can always arrange the blocks into a rectangle.

Use the Lab Gear to model these multiplications. For problems 4-6, sketch the process in two or three steps. (Put on / make a rectangle; or, add zero / take off / make a rectangle.)

4. 2 (–4)
5. –3 (2)
6. –4 (–5)
7. –6 (–2)
8. –10 (2)
9. 7 (–3)

Compute:

10. –9 + 6
11. 2 – (–4)
12. –3 – 2
13. –3 (–2)
14. Copy and complete this sentence: Problem 12 is a subtraction, while problem 13 is a __________.

Multiplying by –1

Use the Lab Gear to model these multiplications.

15. –1 (–2)
16. –1 (3)
17. 4 (–1)
18. –5 (–1)
19. Explain what happens when a number is multiplied by –1.
20. What happens *to the blocks* when you multiply a number by –1? For each example above, sketch the blocks that represent the number *before* and *after* multiplying by –1. Describe what happens in each case.

Squaring Numbers

A special case of multiplication is the multiplication of a number by itself. Use the Lab Gear to model these multiplications. At the end of each problem, check whether the answer is positive or negative, and try to arrange the blocks into a square.

1. $(-3)(-3)$
2. $5 \cdot 5$
3. $(-2)(-2)$
4. $(-6)(-6)$

Multiplying a number by itself is called *squaring* the number. Instead of writing $5 \cdot 5$, you can write 5^2, which is read *five squared.* Instead of writing $(-5)(-5)$, you can write $(-5)^2$. This is read *negative five*, *squared*, or perhaps more clearly, *the square of negative five.* The parentheses tell us that we are squaring –5. The answer is 25.

Be careful! If you write -5^2, without the parentheses, that will be read as *the opposite of five squared*, in other words, the opposite of 25, or –25. In conclusion, $(-5)^2$ and -5^2 are not the same! In fact, they are opposite. If you want to indicate squaring –5, you must use parentheses.

Write how to read each of the following expressions.

5. 2^2
6. $(-4)^2$
7. -3^2
8. $(3x)^2$
9. $(-3x)^2$
10. $-3x^2$
11. In problems 8, 9, and 10, which expressions are equal to each other? Explain your answer. Hint: Work out the numerical value for each one if $x = 2$, and then if $x = -2$.

When writing 5^2, the 2 is called the *exponent*, and the 5 is called the *base*. When writing $6x^2$, the 6 is called the *coefficient*.

12. In the expression 7^2, what do you call the 2? the 7?
13. In the expression $3xy$, what do you call the 3?

Exploration 1 Positive or Negative?

Look at these expressions. For each one, indicate with 0, P, and/or N, if the value of the quantity is zero, positive, or negative. If it is impossible to tell, write ?.

1. 2^2
2. $(-4)^2$
3. -3^2
4. $(3x)^2$
5. $(-3x)^2$
6. $-3x^2$

Generalizing

While using the Lab Gear to do integer arithmetic, you may have discovered some rules, tricks, or patterns that will help you deal with minus signs when doing arithmetic without the blocks. Write down any such rules or patterns.

1. Addition rules, tricks, or patterns
2. Subtraction rules, tricks, or patterns
3. Multiplication rules, tricks, or patterns
4. Squaring rules, tricks, or patterns

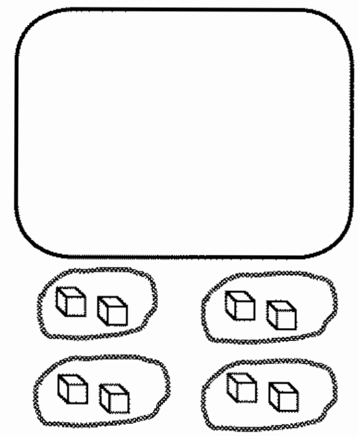
You have been working with integers. However, the rules for integer arithmetic still apply when working with decimals and fractions. For example, the square of a negative fraction, just like the square of a negative integer, is always positive. Try to do the following computations without pencil or calculator:

5. $-5.2 + 11.36$
6. $-2.2 - 0.06$
7. $-3.07\ (1000)$
8. $-\frac{2}{3} + \frac{1}{6}$
9. $-\frac{2}{3} - \frac{1}{6}$
10. $-\frac{2}{3} \cdot \frac{1}{6}$

Division

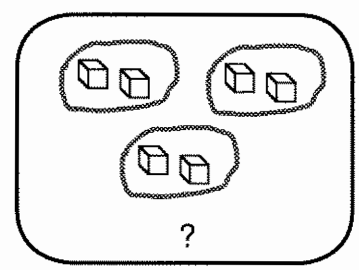
You may be wondering how division is performed with the Lab Gear. Later, you will learn to use the Lab Gear to divide some expressions that involve variables. For integers, the best method is to use the fact that division is the inverse operation of multiplication.

For example, to divide 8 by 2, you could ask, "What times 2 equals 8?" Remember that for multiplication, you start with an empty mat. How many sets of 2 should you *put on* to get 8? It is easy to see that the answer is 4.

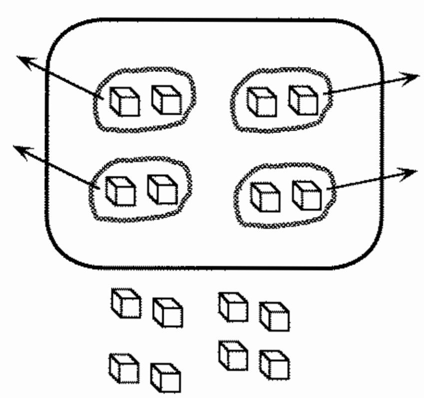


1. Consider the division –8 ÷ –2. What is the multiplication question to ask? How would you solve it on the workmat? Write the answer.

Now consider dividing 8 by –2. Ask yourself, "What times –2 equals 8?" Start with an empty workmat. How many sets of (–2) should you *put on* to get 8? Clearly that question has no answer, since no matter how many times –2 is *put on* you will only get negative numbers.



Try another approach. How many sets of (–2) should you *take off* to get 8? Start with an empty workmat and add zero in the form of 8 and –8. Now try *taking off* four sets of –2 to leave 8. This works. So the answer to "What times –2 equals 8?" is –4. So, 8 ÷ –2 = –4.



2. Consider the division $-8 \div 2$. What is the multiplication question to ask? How would you solve it on the workmat? Write the answer.

3. Using the method shown in the examples, try to figure out rules for division. How do negative numbers in the numerator and/or denominator affect the result? Give examples.

4. Compare these rules to the ones for multiplication.

5. What happens when a number is divided by -1?

6. Which of these divisions are equal?

   a. $\frac{8}{-2}$

   b. $\frac{-8}{2}$

   c. $\frac{-8}{-2}$

   d. $\frac{8}{2}$

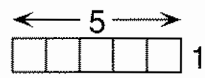
   e. $\frac{-8}{2}$

Exploration 2 Area and Perimeter

The **perimeter** of a two-dimensional figure is the number of units of length around the figure. The unit of length we will use is the centimeter (cm).

The **area** of a figure is the number of square units it would take to cover it. The unit of area we will use is the square centimeter (cm^2). When we discuss the perimeter and area of the Lab Gear blocks, we will be thinking of the tops of the blocks, which are flat, two-dimensional figures.

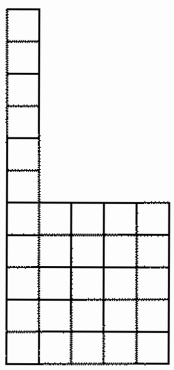
For example, if you look at the 5-block from above, you would see this rectangle. Its area is 5 cm^2, and its perimeter is 12 cm.



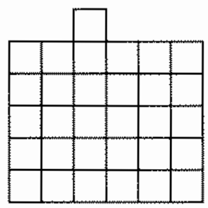
Area: 5 cm^2
Perimeter:
$5 + 1 + 5 + 1 = 12$ cm

Find and write the area and perimeter of these figures, which are the top faces of groups of yellow blocks.

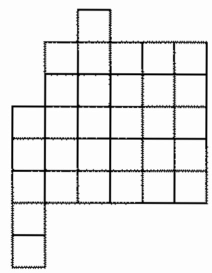
1.



2.



3.



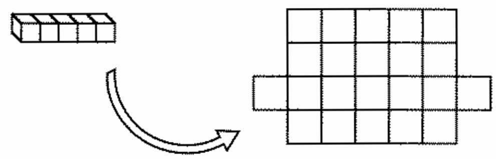
Exploration 3 Volume and Surface Area

The **volume** of a block is the number of cubic units it would take to build it. The unit of volume we will use is the cubic centimeter (cm^3).

For example, the volume of the 5-block is 5 cm^3.

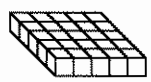
The **surface area** of a block is the number of square units it would take to cover all its faces. To figure out the surface area, it helps to think of a paper jacket that would cover the whole block. The area of such a jacket is the surface area of the block.

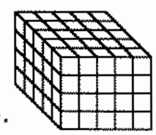
For example, the surface area of the 5-block is 22 cm^2.

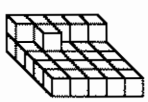


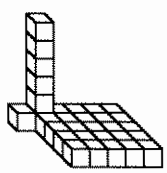
Find and write the volume and surface area of each of these buildings. Notice that in problems about surface area or volume, upstairs blocks do not mean minus.

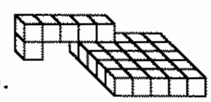
1. 

2. 

3. 

4. 

5. 

6. 

(For the remaining problems in this binder, the units will always be assumed to be cm, cm^2, or cm^3 for length, area, and volume, respectively.)

Chapter 3 Getting to Know the Lab Gear™

This chapter completes the introduction of the Lab Gear and of the basic Lab Gear techniques.

New Words and Concepts

This chapter introduces the use of the corner piece to structure the rectangle interpretation of multiplication and division. It is a crucial part of the Lab Gear which will be used repeatedly throughout the book.

Substituting is an important concept in applications of algebra, for example in the use of formulas in any science. It is also useful in checking the correctness of algebraic manipulations, and in bringing abstract work with variables down to the more concrete level of arithmetic. Doing the substitution with the blocks helps students avoid the "careless" arithmetic errors which are too frequent when doing this sort of manipulation.

The concept of **simplifying** expressions is introduced as well as the notion of **inequalities**.

Teaching Tips

In the long run, accuracy in substituting multi-digit values of the variable into complicated expressions is not a crucial skill. Programmable calculators and some computer software can automate the process, speed it up, and make it error-free. For example, in the Logo language to substitute 12.34 for x in the expression $5.67x^2 - 8.9x + 10.1112$ you can create the following procedure:

```
TO TRINOMIAL :X
OUTPUT 5.67 * :X * :X — 8.9 * :X + 10.1112
END
```

Once that procedure is in the Logo workspace, you can type

```
TRINOMIAL 12.34
```

to perform the desired substitution. In fact, you can rapidly carry out many substitutions. If your students have access to Logo, this is an important use of the language for them to learn.

Note that the shape of the corner piece resembles the symbol for long division, and that the dividend, divisor, and quotient appear in the appropriate positions.

Lesson Notes

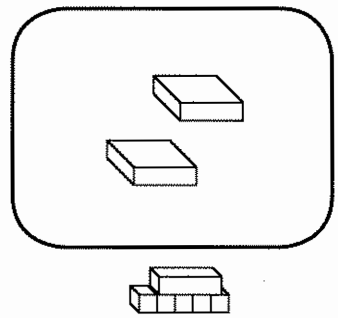
- **Lesson 1,** Substituting, page 30: You may want to demonstrate the substituting of a negative number for x on the overhead projector.
- **Lesson 2,** Using the corner piece, page 32: This is a straightforward example, using positive numbers. Use it as an introduction to the following exploration and lessons.
- **Lesson 3,** Multiplication, page 34: The rectangle model breaks down unexpectedly in some simple cases, such as problems 9 and 10, which cannot really be represented visually in a satisfactory manner—at least not with a rectangle. Hopefully, your students will be able to extrapolate from what they learned, and guess at the correct application of the distributive rule.
- **Lesson 4,** Division, page 35: Instead of dividend and divisor, the words numerator and denominator are used, for two reasons: they are more familiar to most students, and they underline the relationship between fractions and division, which cannot be stressed enough.
- **Lesson 5,** Simplifying, page 36: Solving inequalities provides the students with a reason for simplify the expressions. It is best to avoid simplifying "for its own sake" in unmotivated exercises.
- **Lesson 6,** Which is Greater?, page 39: Do not tell the students to add or subtract blocks from both sides of the inequality. They do not yet need this technique, and it is likely they will discover it for themselves in this lesson or in future "Which is Greater?" explorations. Next one: p.52.
- **Lesson 7,** Parentheses, page 42: Insist that students show both sides of the equation with blocks.
- **Lesson 8,** How Much More? How Many Times as Much?, page 43: This is an optional lesson.

Substituting

To solve these problems, you will need to **substitute** the given constants for the variables, cancel, and count. Be very careful with variables in the minus area, or upstairs.

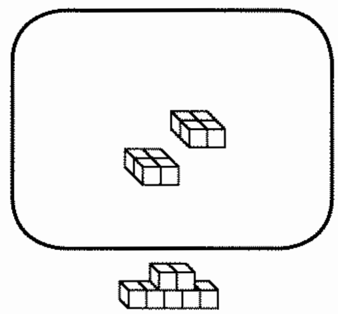
Look at this example.
These blocks show $-2x^2 - x + 5$.
Substitute these numbers for x.

a. 2
b. −3



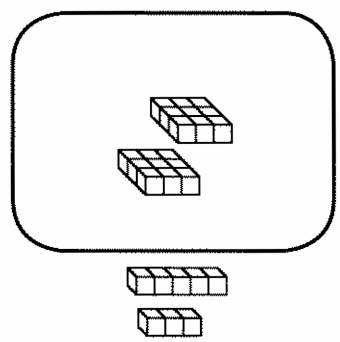
In part a, the substitution is straightforward. Put two 1-blocks in place of the x-block, and 2^2 (or four) 1-blocks in place of each x^2-block.

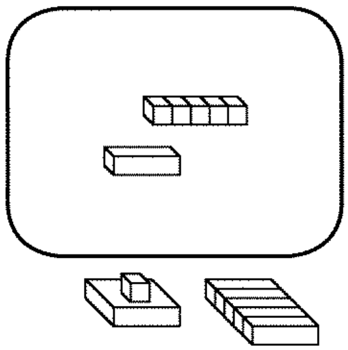
1. Write the final answer for part a.



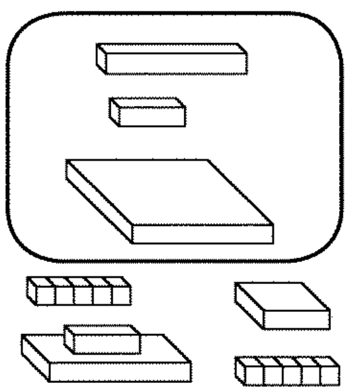
In part b, each x^2-block is replaced by $(-3)^2$, or nine 1-blocks.

2. Why was the x-block, which was upstairs, replaced by three 1-blocks *downstairs*?
3. Write the final answer for part b.



4. 

a. Write the expression shown by the figure above.
b. If $x = 2$, what is x^2? What value do the blocks show?
c. If $x = -3$, what is x^2? What value do the blocks show?
d. If $x = -5$, what value do the blocks show?

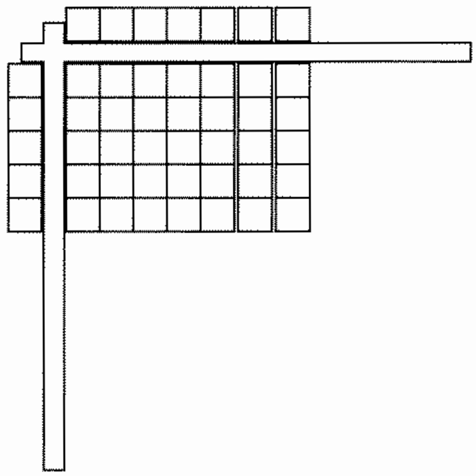
5. 

a. Write the expression shown by the figure above.
b. If $x = 1$ and $y = -2$, what is x^2? What is y^2?
What is xy? What value do the blocks show?
c. If $x = -3$ and $y = 2$, what value do the blocks show?
d. If $x = 5$ and $y = -5$ what value do the blocks show?

Using the Corner Piece

As you know, the result of any multiplication can be shown as a rectangle. This is because for a rectangle, **area = length · width**.

Arrange your corner piece and blocks to match this figure.

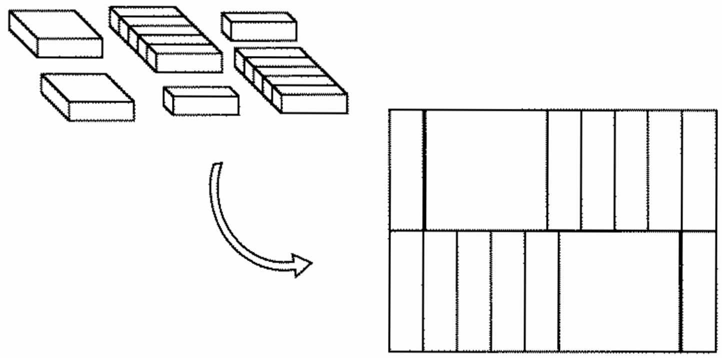


1. Write the multiplication equation that is shown by this figure.
2. Write a division equation that is shown by this figure.
   (area / length = width)
3. Write another division equation that is shown by this figure.
   (area / width = length)
4. Use the corner piece to set up as many different divisions as you can with numerator 12. Sketch each one. For each, write the division equation and the corresponding multiplication equation.
5. Explain why it is impossible to set up the division $\frac{12}{0}$ with the blocks.
6. Some algebra students believe that $\frac{12}{0} = 0$. Explain why they are wrong by discussing the multiplication that would correspond to this division.

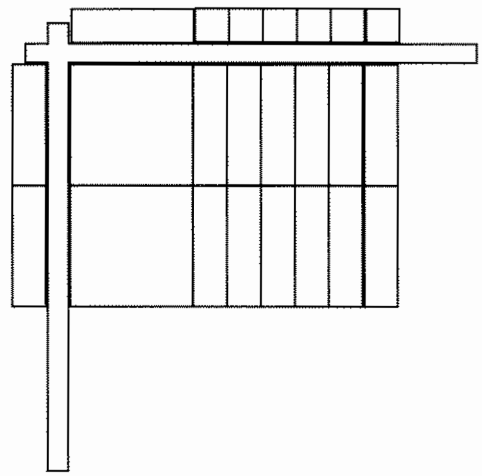
Exploration 1 Make a Rectangle

Sometimes a group of blocks can be arranged into a rectangle or square.

For example, you can rearrange the blocks $2x^2 + 12x$ into a rectangle like this.



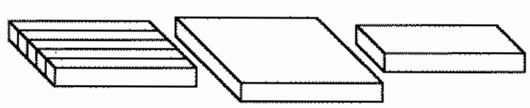
The length and width of this rectangle are $x + 6$ and $2x$, which can be seen better if you organize the blocks logically and use the corner piece, as shown. (Notice that you could also turn the rectangle so that the length and width are exchanged. This is considered to be the same rectangle.)



1. There is another rectangular arrangement of the same blocks which has different dimensions. Find it.

For each problem, arrange the given blocks into a rectangle. Sketch it (as seen from above) and write the length and width. Problem 4 has two solutions (find them both).

2. 

3. 

4. 

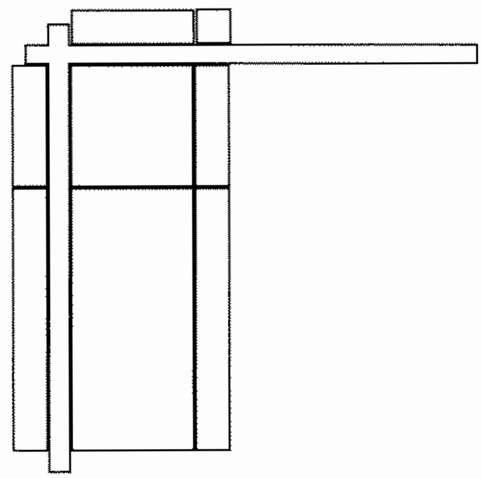
For each problem, try to guess the dimensions of a rectangle that could be made with the blocks. Then check your answer with the Lab Gear. One is impossible. Explain why.

5. $3x^2 + 9x$
6. $3xy + 2y + y^2$
7. $4x^2 + 9y$
8. $x^2 + 5x$

Multiplication

Arrange your corner piece and blocks to match this figure.

1. Write a multiplication and a division equation for the figure.



The multiplication equation one student suggested was $(x + y)(x + 1) = x^2 + x + xy + y$. To check whether this is correct, try substituting 3 for x and 2 for y on both sides of the equal sign. If you get a true statement, that will help convince you that the equality was valid.

$$(3 + 2)(3 + 1) \stackrel{?}{=} 3^2 + 3 + (3 \cdot 2) + 2$$
$$5 \cdot 4 \stackrel{?}{=} 9 + 3 + 6 + 2$$
$$20 = 20$$

2. Check again, by substituting other numbers for x and y.

Use the corner piece and your blocks to show these multiplications. Write the products.

3. $(x + 1)\,x$
4. $(x + 2)(x + 3)$
5. $(x + 5)(x + y)$
6. $2y\,(y + 1)$
7. $(y + 4)(y + 1)$
8. $(2x + 3)(x + y + 1)$

These multiplications cannot be modeled using the corner piece. Try to find the products another way.

9. $3\,(4x^2 + 6)$
10. $5\,(y^2 + 2y + 9)$

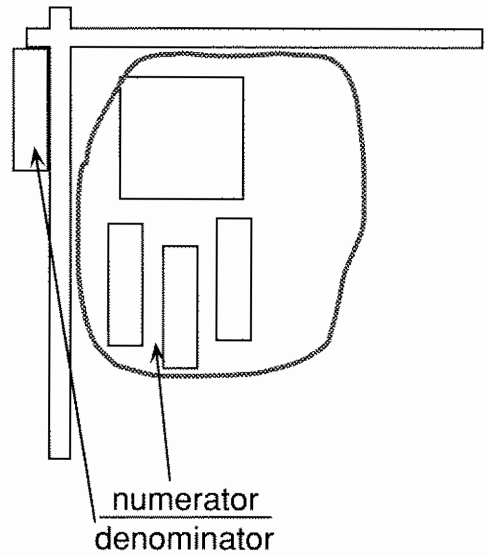
☑ Self-check

4. $x^2 + 5x + 6$
9. $12x^2 + 18$

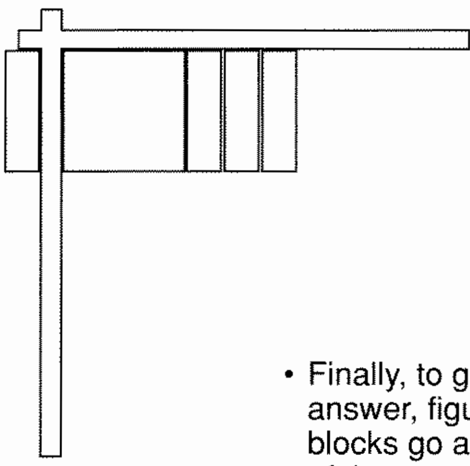
Division

Use your corner piece and blocks to follow this method for solving division problems.

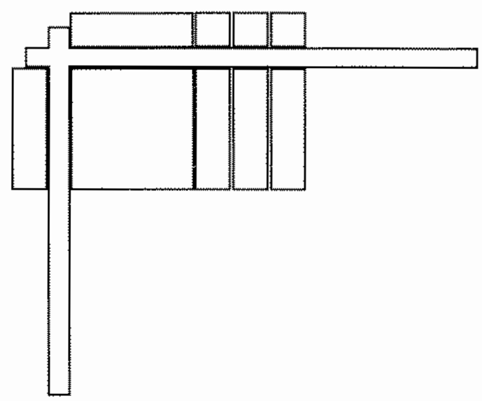
- Put the denominator to the left of the corner piece.



- Make a rectangle out of the numerator, and place it inside the corner piece so that one side of the rectangle matches the denominator.



- Finally, to get the answer, figure out what blocks go along the top of the corner piece.



1. Write the division equation shown by this figure.

Use this method to solve these division equations. Write the related multiplication equation. Not all are possible to solve with blocks.

2. $\frac{x^2 + 4x}{x} =$
3. $\frac{6x + 9}{3} =$
4. $\frac{x^2 + xy}{x} =$
5. $\frac{6x^2 + 3x}{3x} =$
6. $\frac{3y + 2}{5} =$
7. $\frac{4y^2 + 6y}{2y} =$
8. $\frac{2y^2 + y}{y} =$
9. $\frac{4y + 2x + 10}{2} =$

These division equations cannot be modeled clearly with the corner piece. Try to simplify them another way.

10. $\frac{6x^2 - 9}{3} =$
11. $\frac{4x^2 + 6x - 10}{2} =$

☑ Self-check

10. $2x^2 - 3$

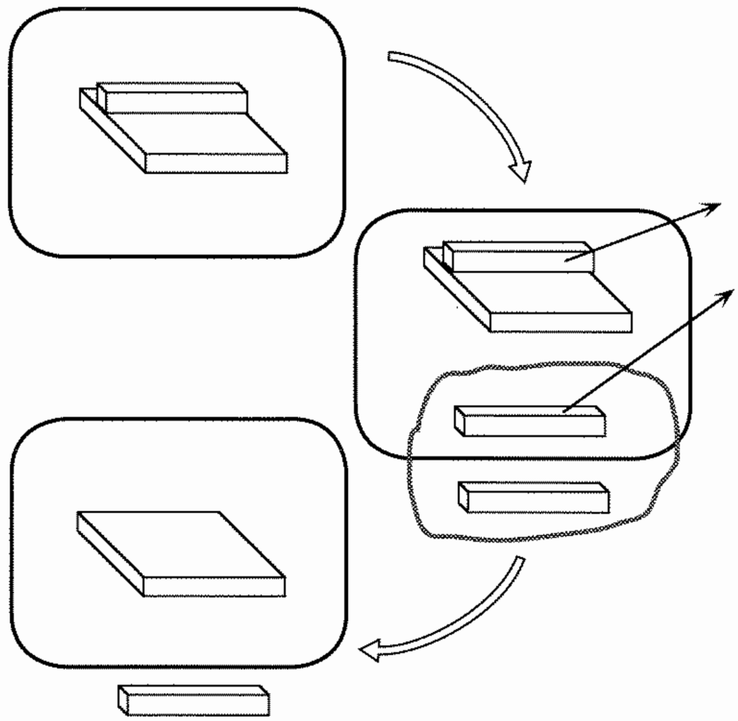
Simplifying

Here is a technique you will need to know. To simplify upstairs blocks in the minus area, you can use the adding zero trick.

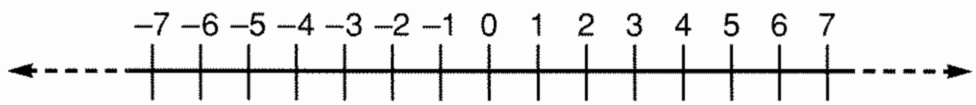
For example, this figure shows how to simplify $-(y^2 - y)$.

- Add the same quantity downstairs inside and outside of the minus area.
- Cancel matching upstairs and downstairs blocks.

The simplified form is $y - y^2$.



Which is Greater?

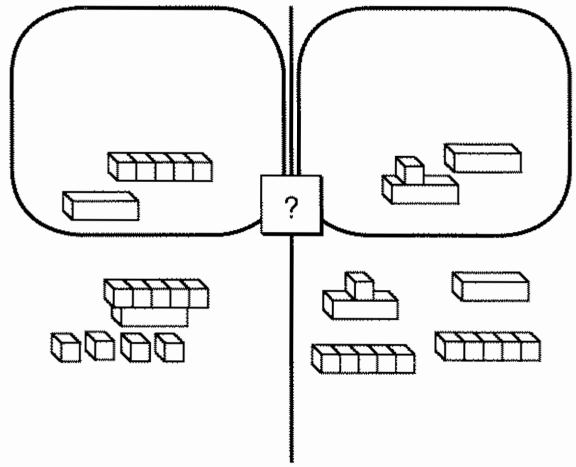


You can tell which of two numbers is greater by its position on the number line. The number that is greater is farther to the right. The number that is less is farther to the left. The symbol for *less than* is $<$. For example, $-5 < 3$, $0 < 7$, and $-6 < -2$. The symbol for *greater than* is $>$. For example, $6 > 3$, $0 > -2$, and $-5 > -9$.

1. Use the correct symbol.
   a. -5 **?** -7 b. -5 **?** -1

This workmat shows two expressions:
$x + 4 - 5 - (x + 5)$ and
$10 + 2x - 1 - (2x - 1)$.

Which is greater? The question mark shows that this is unknown.

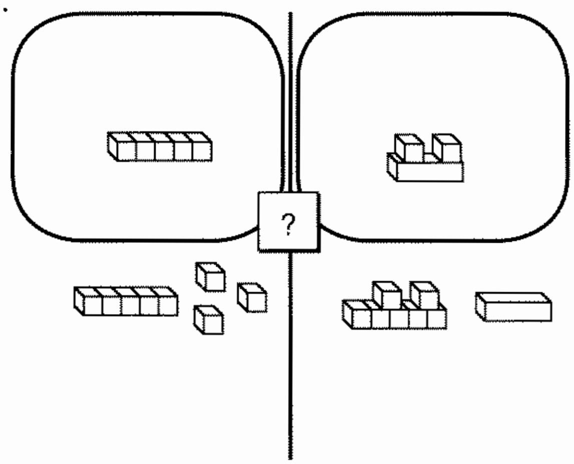


2. Mark the blocks that can be cancelled. Look at the blocks that remain. Write an expression for the blocks on the left side. Write an expression for the blocks on the right side. Which side is greater? Show your answer by writing the correct **inequality sign** between the two expressions.

For each problem, put out blocks to match the figure, and:

a. Write the two expressions.
b. Simplify both sides on the workmat.
c. Decide which side is greater or whether they are equal and write the correct sign between the expressions.

3.



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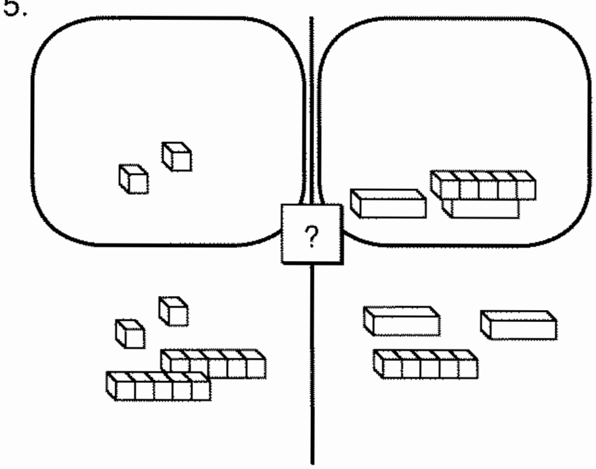

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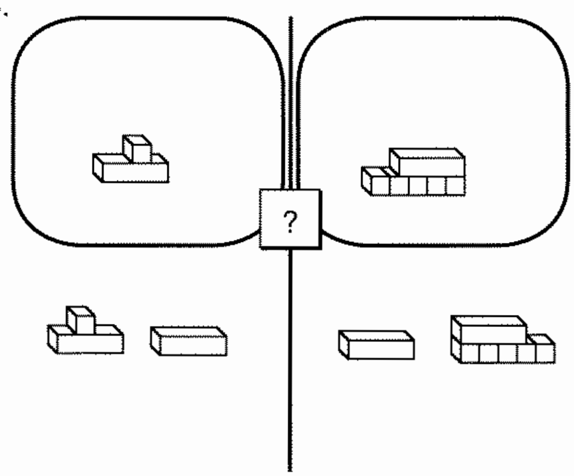

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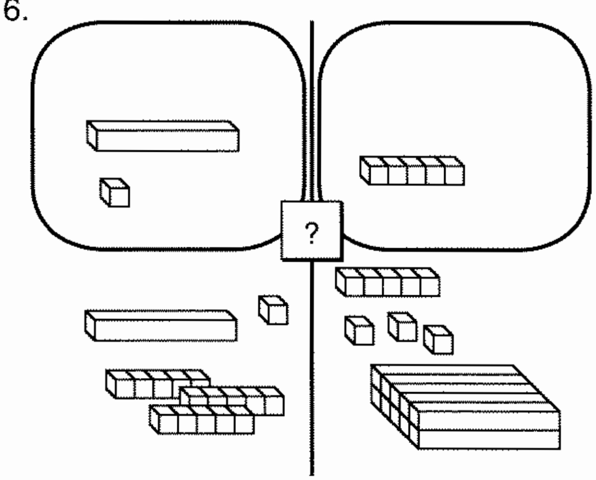
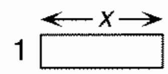


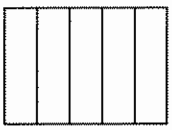
Exploration 2 Area and Perimeter

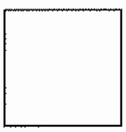
To determine the area and perimeter of the variable blocks, we will not use the actual measurements. Instead, we will consider their dimensions in terms of x and y.

For example, this figure, the top of an x-block, is a 1 by x rectangle. So its area is $1 \cdot x = x$, and its perimeter is $x + 1 + x + 1$, which, by combining like terms, can be written $2x + 2$.

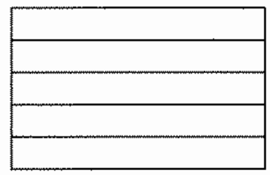


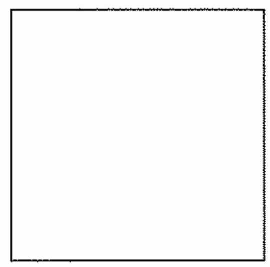
Find and write the area and perimeter of the following rectangles, which are the top faces of the remaining variable blocks. Be careful when collecting like terms.

1. 

2. 

3. 

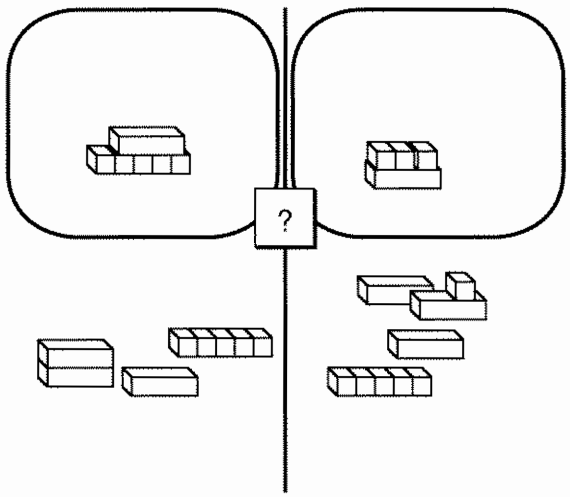
4. 

5. 

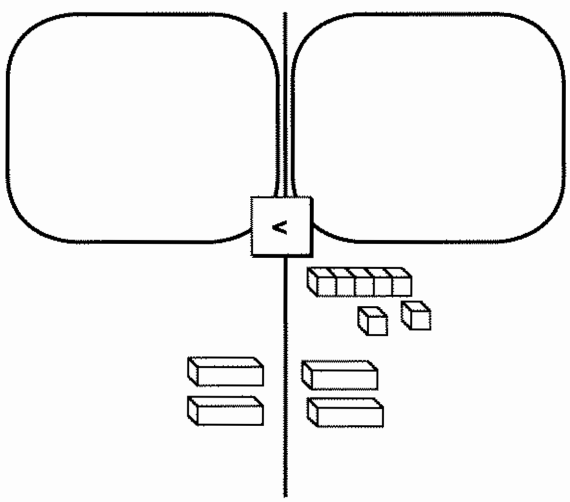
6. 

Which is Greater?

To compare $2x - x + 5 - (5 - x)$ with $5 + 3x - 1 - (x - 3)$, first show the two expressions with the Lab Gear.

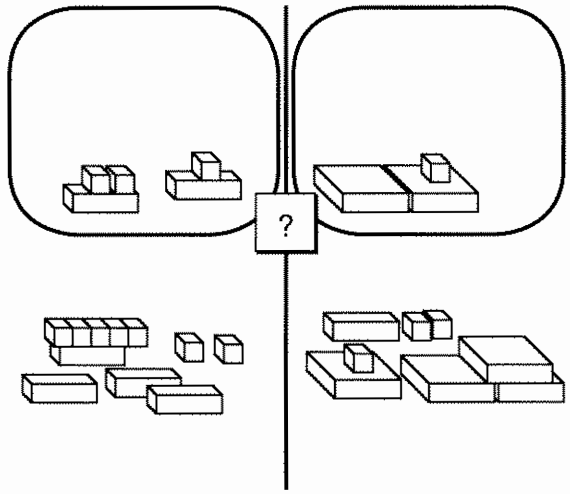


Simplify both sides and arrange the blocks in a logical manner to determine which side is greater. Both sides include $2x$, but the right side is greater as it also includes 7 more units.



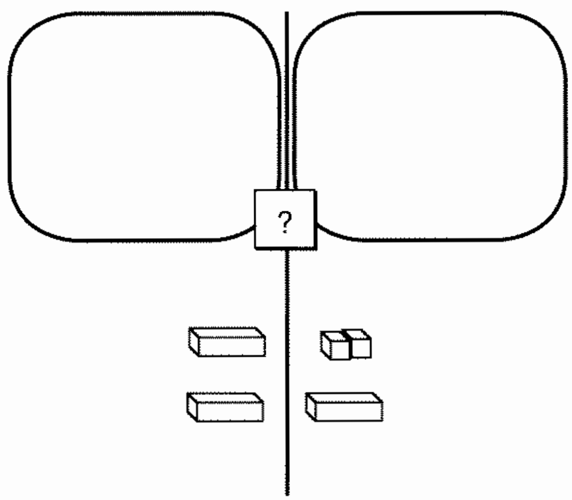
Now compare these expressions.

1. Write both expressions as they are shown in this figure.



2. Mark the blocks that can be cancelled. Now write the simplified expressions.

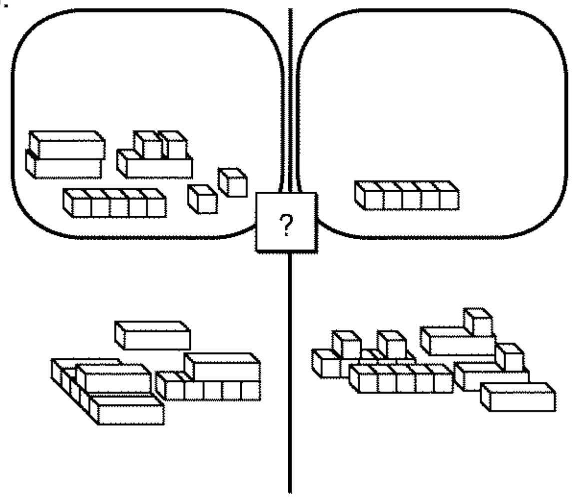
Your workmat should look like this.



In this case, it is impossible to tell which side is greater, because we do not know whether x is greater or less than 2.

For each of these problems, simplify using your blocks, and write the expressions in simplified form. Decide which side is greater, or whether they are equal, or whether it is impossible to tell. Write the correct symbol or "?".

3.



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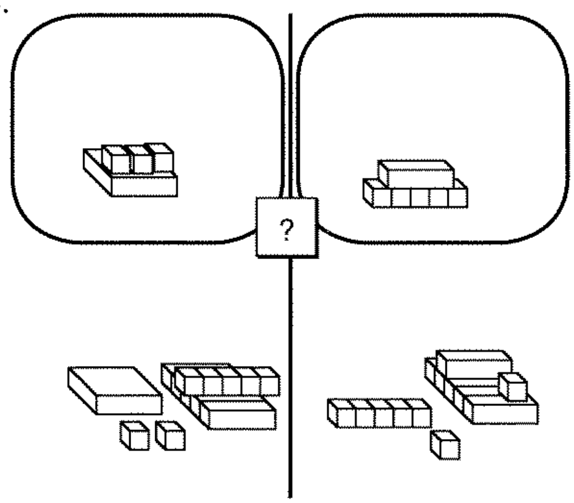

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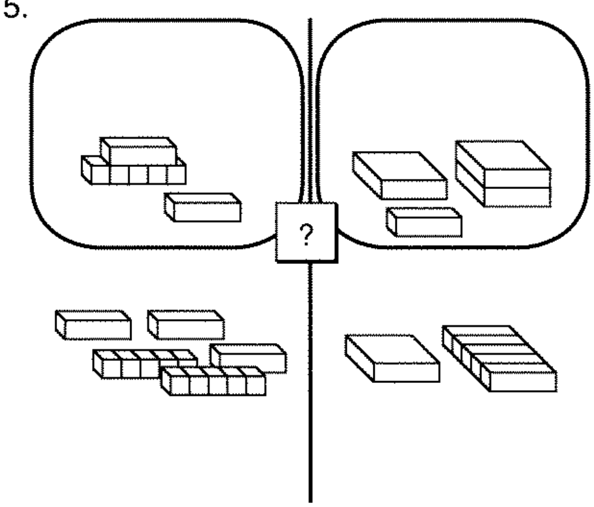

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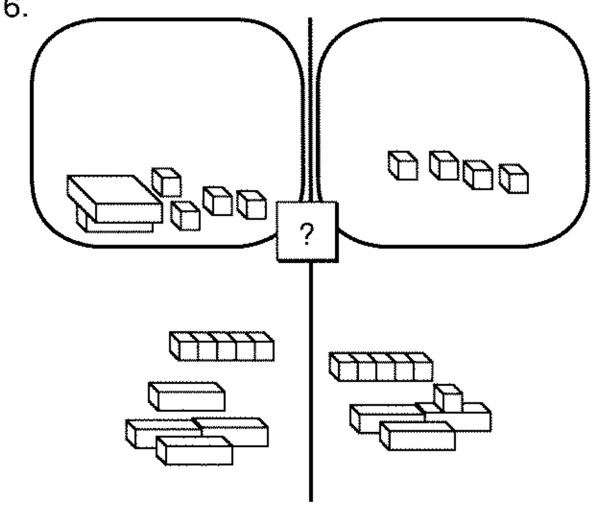


Parentheses

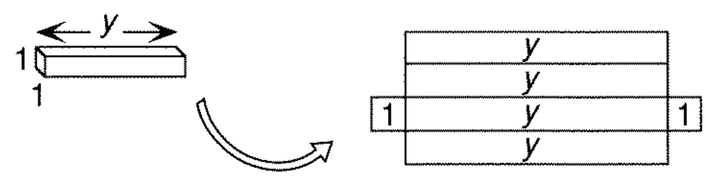
Use the Lab Gear to help you decide which of the expressions a, b, c, or d are equal to the expression on the left. Explain your answers.

1. $-(x+y)$	a. $-x+(-y)$	b. $-x-y$	c. $-x+y$	d. $y-x$
2. $-(x-y)$	a. $-x+y$	b. $-x-y$	c. $-x-(-y)$	d. $y-x$
3. $-(-x+y)$	a. $-x+y$	b. $-y-x$	c. $x-y$	d. $y-x$
4. $-(y-x)$	a. $x-y$	b. $-x+y$	c. $-y+x$	d. $-y-(-x)$

Exploration 3 Volume and Surface Area

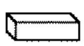
The surface area of the variable blocks can be figured out by thinking of their jackets.

For example, the y-block has surface area of $4y+2$.



Its volume can be found by multiplying *length* times *width* times *height*. $1 \cdot 1 \cdot y = y$.

Find and write the volume and surface area of the remaining variable blocks.

1. 
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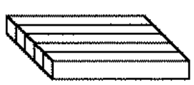
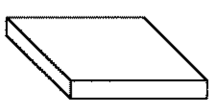
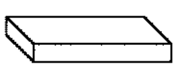
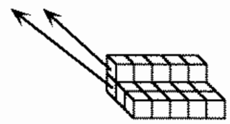
4. 
5. 
6. 

How Much More? How Many Times as Much?

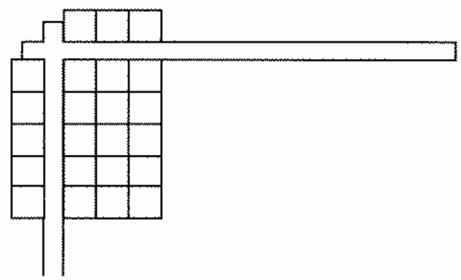
When comparing the size of two positive numbers, for example 5 and 15, you can ask two different questions.

- 15 is *how much more* than 5?
- 15 is *how many times greater* than 5?

Using the Lab Gear, the first question is answered like this. So, 15 is 10 more than 5.



The second question is answered like this. So, 15 is 3 times greater than 5.



Ask and answer both questions about these pairs of numbers. The Lab Gear may help you answer.

1. 25 and 5
2. 6 and 1
3. 4 and 2
4. 15 and 3
5. 42 and 7
6. 10 and 10
7. 9 and 8

Exploration 4 Always, Sometimes, or Never?

If a statement does not include any variables, it is either true or false. For example, $7 = 5 + 2$ is true, while $9 \cdot 7 = 54$ is false.

But if a statement contains one or more variables, there are three possibilities.

- Some statements are *always true*. For example, $x + 2 = 2 + x$. You can substitute any number for x, and you will get a true statement.
- Some statements are *never true*. For example, $x + 2 = 3 + x$. You can substitute any number for x, and you will get a false statement.
- Some statements are *sometimes true*. For example, $x^2 = 2x$. If you substitute a number for x, you will probably get a false statement. (Try 3, or 1, or –10.) But if you substitute 2 or 0 for x, you get a true statement.

Write A, S, or N for each of the following statements, depending on whether it is always, sometimes, or never true. Use the Lab Gear to help you decide.

1. $-(x-4) = -x + 4$
2. $x + 4x + x = 6x$
3. $2 + x = 2x$
4. $y(y + 2) = y^2 + 2$
5. $2x + 5 = 2x + 1$
6. $\dfrac{x^2 + 2x}{x} = x + 2$
7. $y^2 - y - 1 = y^2 - y - 5$
8. $x + 15 = 2x$

Chapter 4 Simplifying Algebraic Expressions

The main purpose of this chapter is to lead the students in a discovery of the correct handling of addition and subtraction of expressions that include variables.

New Words and Concepts

This chapter focuses on adding and subtracting **monomials** and **polynomials**, in particular the distinction between parentheses that are preceded by a plus sign and parentheses that are preceded by a minus sign. (The full treatment of parentheses that are preceded by a multiplication symbol is left for Chapter 5.) With the Lab Gear, combining "unlike" terms is a very unusual mistake. The students practically teach themselves that only "like" terms can be combined.

Teaching Tips

Encourage students who discover good techniques for finding the perimeters and surface areas of given figures to share them with their classmates. The best strategy is to use subtraction in order to find the length of a side, or the area of a face where two blocks make contact.

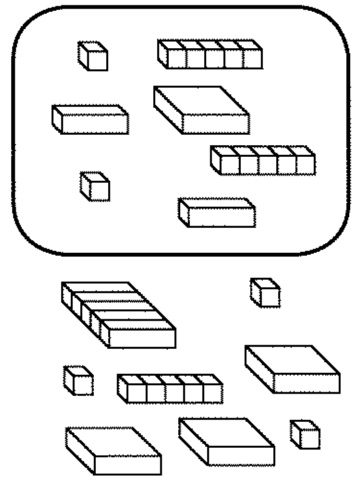
Lesson Notes

- **Lesson 1,** Polynomials, page 45: You may need to have some students use the overhead projector to remind the class how to simplify upstairs blocks in the minus area. See Chapter 3, Lesson 5.
- **Lesson 2,** Removing Parentheses, page 47: Insist on seeing each equation represented with blocks. It is, of course, faster to work out the arithmetic than to set up the blocks, but working out the visual layout of each expression will guarantee a deeper understanding.
- **Lesson 3,** Adding Polynomials, page 48: This should be quite straightforward, again a review of combining like terms. If your students really understand this, only assign a few problems from this lesson.
- **Lesson 4,** Subtracting Polynomials, page 49: Make sure the students understand both methods ("taking away" and "adding the opposite").
- **Lesson 5,** More on Parentheses, page 51: Removing parentheses incorrectly is a source of many errors. The rule that this lesson leads to should be memorized. But just knowing the rule is not enough. Insist that the students be able to explain their answers to the problems in this lesson in terms of the blocks.

Polynomials

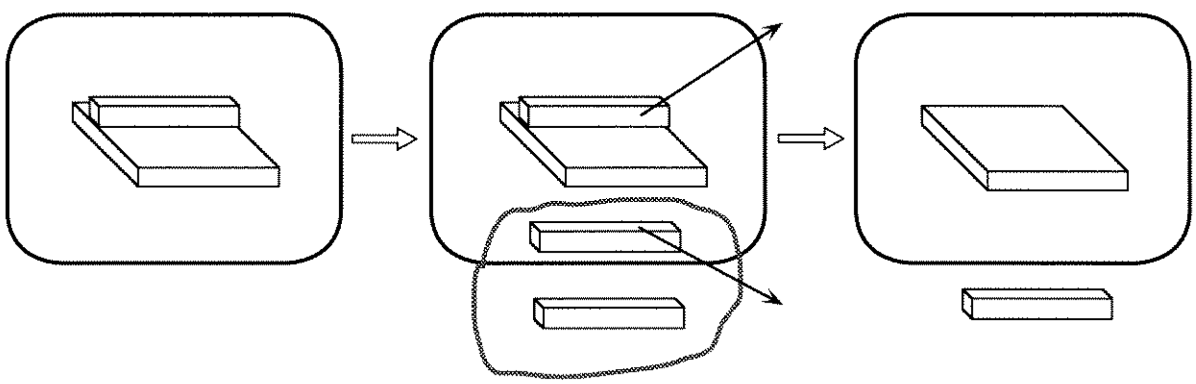
Algebraic expressions like $2x$, or $-3x^3$, or $2xy$ are called **monomials**. Sums of monomials are called **polynomials**.

1. Write the quantity that this figure shows.
2. What quantity does it show if $x = 2$? Use substitution to find out. Replace each x-block with 2. What should you replace the x^2-block with? What should you replace the $5x$-block with? Cancel what you can and write the answer.
3. While working the above problem, you may have thought about *simplifying* the expression *before* doing the substituting. For example, the x^2 in the minus area can be cancelled along with one of the x^2 outside. What else can be cancelled? Shade the blocks that can be removed, and write the simplified polynomial.



4. Use the simplified expression from problem 3. What does it show if $x = 2$? Use substitution to find out, and write your answer. This should be easier to do, and you should get the same answer. (If not, find your mistake!)

As you know, to simplify upstairs blocks in the minus area, you can use the adding zero trick.

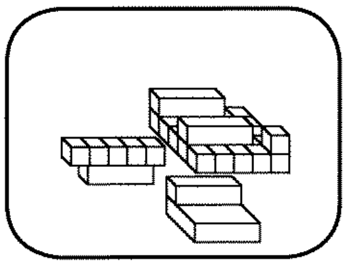


5. Write the quantity shown in this figure before and after it is simplified.

For each figure:

a. Copy the figure with your blocks.
b. Simplify the figure using the blocks.
c. Write the polynomial that names the original blocks.
d. Write the simplified polynomial.

6.



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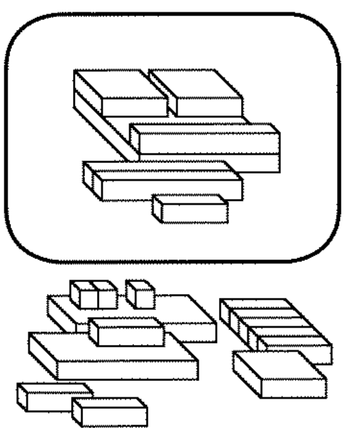

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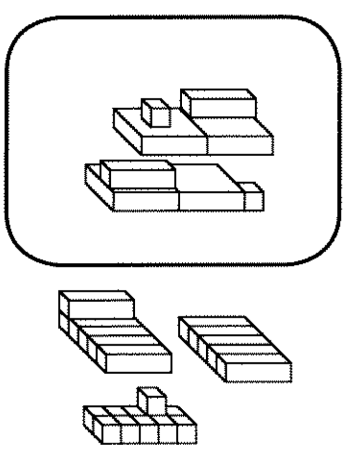

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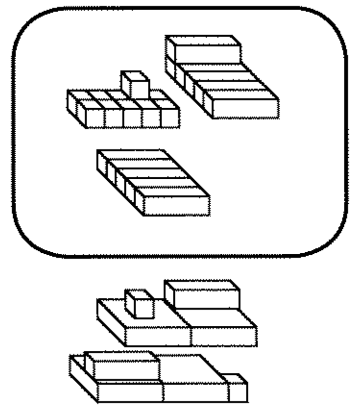

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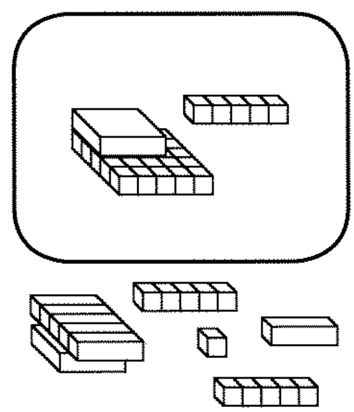


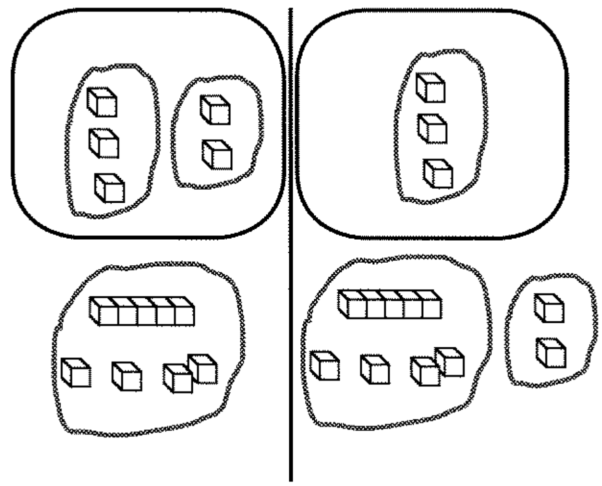
Removing Parentheses

When an algebraic expression is enclosed in parentheses, you should think of it as one unit.

1. For example, if you write 9 – (3 + 2), you are saying *subtract the quantity 3 + 2 from the number 9*. Write the result.

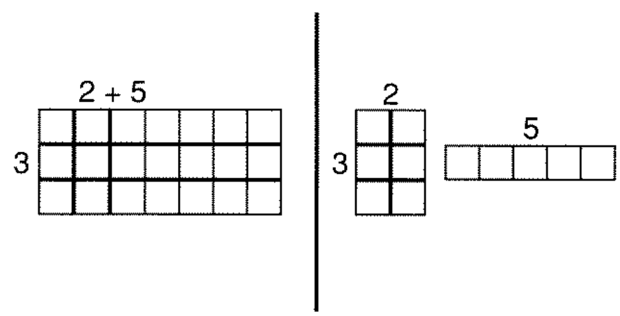
Be careful! If you remove parentheses from an expression, you may change its value without intending to.

2. For example, this figure compares 9 – (3 + 2) and 9 – 3 + 2. Are the two sides equal? Use blocks and simplify to find out.



$9 - (3 + 2) \stackrel{?}{=} 9 - 3 + 2$

For these problems, use the Lab Gear to help you decide if the two sides are equal. Write T if they are equal, F if they are not equal. The figure for problem 3 is shown to get you started.



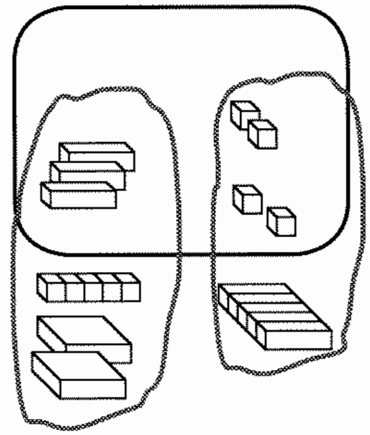
3. $3(2 + 5) = 3 \cdot 2 + 5$
4. $4 + (6 - 2) = 4 + 6 - 2$
5. $4 - (1 + 6) = 4 - 1 + 6$
6. $8 - (3 - 5) = 8 - 3 - 5$
7. $6 - (4 - 3) = 6 - 4 - 3$
8. $2(4 - 7) = 2 \cdot 4 - 7$
9. $8 - (2 + 5) = 8 - 2 + 5$
10. $10 + (3 + 8) = 10 + 3 + 8$
11. $5(-5 + 2) = 5(-5) + 2$
12. $-2(4 - 3) = -2 \cdot 4 - 3$
13. $5(4 - 2) = 5 \cdot 4 - 2$
14. $3 + (2 - 4) = 3 + 2 - 4$
15. Copy and complete this sentence:

    Parentheses that are preceded by ____________ can be removed without changing the value of an expression.

    Check whether you are right by creating more problems of this type. Discuss them with a classmate.

Adding Polynomials

This figure shows $(2x^2 - 3x + 5) + (5x - 4)$



1. Write the sum. Don't forget to cancel what you can.

Use the Lab Gear to add these polynomials.

2. $(x^2 - 4x + 2) + (6x + 5)$
3. $(2y^2 + 3y + 25) + (2y - 10)$
4. $(-y^2 + 2x^2 - 3y) + (y + 5x - x^2)$
5. $(4x^2 - 2xy + 10) + (y^2 - 2x^2 + 3xy + 5y)$
6. $(2x^2 - 2x - 1 + 3xy) + (2y^2 - 3x + 6 + x^2 - xy)$
7. $(x^2 + 7x + 6) + (3x^2 + 2x - 5) + (-x^2 - 3x + 4)$

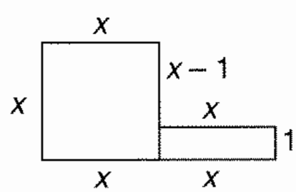
☑ Self-check

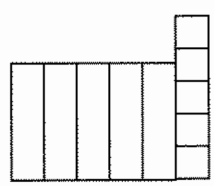
2. $x^2 + 2x + 7$
6. $3x^2 + 2y^2 + 2xy - 5x + 5$

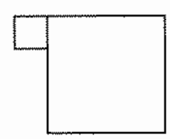
Exploration 1 Perimeter

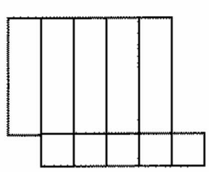
In problems about perimeter, area, and volume, assume that x and y are positive. In fact, assume that $1 < x < 5 < y < 10$.

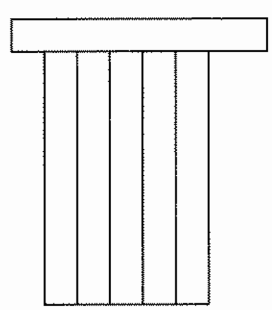
Find the perimeter of these figures.

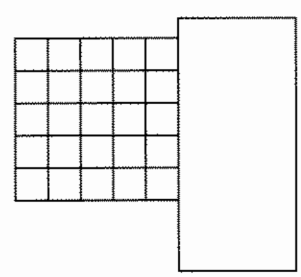
1. 

2. 

3. 

4. 

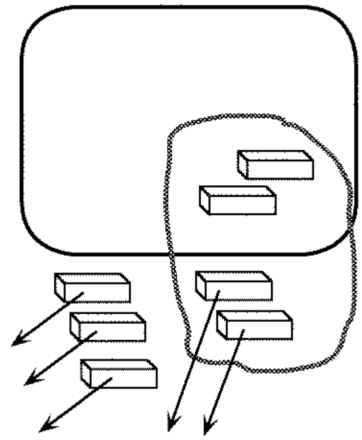
5. 

6. 

Subtracting Polynomials

To calculate $3x - 5x$, you can *put on* $3x$, *add zero*, and *take off* $5x$.

1. Write the answer to this problem.

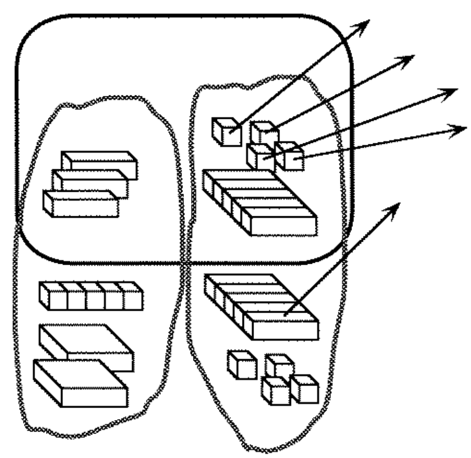


Practice with these problems. Not all of them can be simplified.

2. $2y - 7y$
3. $3xy - (-2xy)$
4. $x^2 - 4x^2$
5. $2xy - 2x$
6. $-5x - 4x$

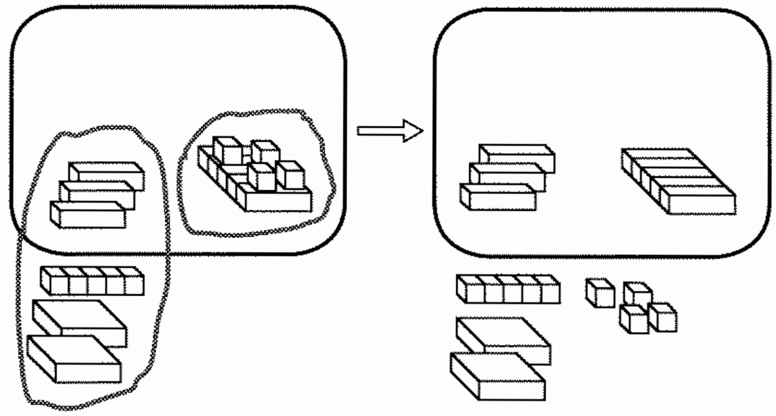
To calculate $(2x^2 - 3x + 5) - (5x - 4)$, you can *put on* the first polynomial, and *take off* the second, after *adding zero* as necessary.

7. Explain the figure. Write the answer.



Look at $(2x^2 - 3x + 5) - (5x - 4)$ again. Another approach to subtracting is to put the whole quantity $5x - 4$ inside the minus area and simplify.

8. Explain what was done to simplify this expression.



9. Copy and complete this sentence: With both methods, to subtract $(5x - 4)$, we ended up adding ____________.

Use the Lab Gear to solve these subtraction problems.

10. $(x^2 - 6x + 4) - (2x - 3)$

11. $(2y^2 + 5y - 2) - (y + 8)$

12. $(y^2 - 3y - 7) - (y^2 + 2y - 3)$

13. $(-4x^2 + 8x - 25) - (3x^2 - 20)$

14. $(2xy + y^2 + 6) - (4y^2 - xy + 5 - x)$

15. $(3x^2 - 2x - 1 + xy) - (2y^2 - 3x + 6 + x^2 - xy)$

16. $(4x^2 - 2xy + 10) - (y^2 - 2x^2 + 3xy + 5y)$

17. $(-y^2 + x^2 - 3y) - (-2x^2 + y - 2y^2 + 5)$

18. $(x^2 + 5x + 6) - (3x^2 + 2x - 7) - (-2x^2 - 3x + 4)$

☑ Self-check

10. $x^2 - 8x + 7$
16. $6x^2 - y^2 - 5xy - 5y + 10$

More on Parentheses

Use the Lab Gear to help you decide which of the expressions a, b, c, or d are equal to the expression on the left. Explain your answers.

1. $x-(5+2x)$
   a. $x-5+2x$
   b. $x-5-2x$
   c. $x+5+2x$
   d. $x+5-2x$

2. $y-(6+3y)$
   a. $y-6+3y$
   b. $y-6-3y$
   c. $y+6+3y$
   d. $y+6-3y$

3. $2x-(-4+3x)$
   a. $2x-4+3x$
   b. $2x-4-3x$
   c. $2x+4+3x$
   d. $2x+4-3x$

4. $3y+(5-2y)$
   a. $3y-5+2y$
   b. $3y-5-2y$
   c. $3y+5+2y$
   d. $3y+5-2y$

5. $x-(7-2y)$
   a. $x-7+2y$
   b. $x-7-2y$
   c. $x+7+2y$
   d. $x+7-2y$

6. $6x-(-3-x)$
   a. $6x-3+x$
   b. $6x-3-x$
   c. $6x+3+x$
   d. $6x+3-x$

Find a way to write an equivalent expression without parentheses.

7. $2x^2-(4-x-x^2)$

8. $(2x^2-4)-(x-x^2)$

9. Explain how you can correctly remove parentheses from an algebraic expression when there is a minus sign before the parentheses.

Exploration 2 Which is Greater?

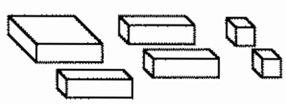
For each of these problems, represent both expressions with the Lab Gear. Simplify the two expressions, and then decide which side is greater, whether they are equal, or whether it is impossible to tell. Write =, >, <, or ?.

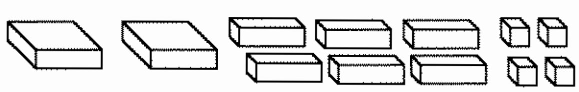
Which is greater…

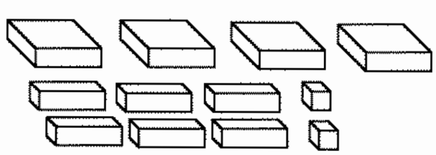
1. $5x + x^2 + 1 - 2 - (2x^2 + 2x - (x^2 + x + 1))$ or $3x + 3 - 5 - (2x - (x + 2))$?
2. $4 + 10x - 2x - (3x + 2 - 1)$ or $5x + x^2 - 1 - (x + x^2 + 4 - x)$?
3. $7x - x^2 - (2x - (x + x^2))$ or $6x + 5 + x^2 - (x^2 + 5 - (2x^2 + 3))$?
4. $6x + 5 - (2x + 2) - (2x + 5 - 2)$ or $2x + 1 + x^2 - 4 - (x^2 - 3)$?
5. $5x + 8 - 2 - (3x + 5 - 2)$ or $3x + 7 + x^2 - 1 - (4 + x^2)$?
6. $10x + 2 - 5 - (3x + 1 - 4)$ or $7x + x^2 - x - (3x^2 - (2x^2 + x))$?

Exploration 3 Make a Rectangle

For each problem, arrange the given blocks into a rectangle or square. Sketch it (as seen from above) and write a multiplication equation relating the length, width, and area of the figure. Some have two solutions (find them both).

1. 

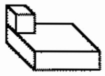
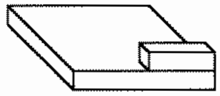
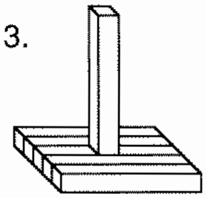
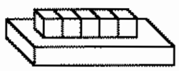
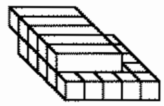
2. 

3. 

4. $x^2 + xy + x + y$
5. $x^2 + 7x + 10$
6. $3x^2 + 5x + 2$
7. $6x^2 + 7x + 2$
8. $6x^2 + 19x + 10$
9. $3x^2 + 16x + 5$
10. $4x^2 + 8x + 4$

Exploration 4 Surface Area

Imagine these buildings are made by gluing Lab Gear blocks together. The surface area is the total area of all the exposed faces, even the bottom of the building. Find the surface areas.

1. 
2. 
3. 
4. 
5. 

Exploration 5 Always, Sometimes, or Never?

Write A, S, or N for each of these statements, depending on whether it is always, sometimes, or never true. Use the Lab Gear to help you decide. Explain your answers.

1. $x(3 + 2) = x \cdot 3 + 2$
2. $x^2 - (2 + 7) = x^2 - 2 + 7$
3. $y(y - 3) = y^2 - 3$
4. $5 + (3 - x) = 5 + 3 - x$
5. $6 - (y - 4) = 6 - y - 4$
6. $4 + (x + 2) = 4 + x + 2$

Chapter 5 Multiplying and Dividing

This chapter focuses on the multiplication and division of expressions that include variables.

New Words and Concepts

The **distributive law** may be the most crucial idea of elementary algebra. By now, especially if they have done the "Make a Rectangle" Explorations, your students should have the intuitive foundation on which to build both understanding of the rule, and proficiency in its application. Note that a Lab Gear version of the law is presented first: "Multiply every block across the top by every block along the side." Then the students are led to discover the usual formulation of the law through their work with the blocks.

The "foil" method for multiplying binomials, while popular with students, is not presented in this book. The reason is that it does not readily generalize to other cases, such as multiplication of a binomial by a trinomial. The formulation: "Multiply every term in the first polynomial by every term in the second polynomial" (which will be stated in Chapter 6) applies with absolute generality to every polynomial multiplication, and is a natural extension of the work with the corner piece.

Cubes and other third degree monomials are introduced, though they will not be used further.

The symbols for **less than or equal to** and **greater than or equal to** are first used in this chapter.

Teaching Tips

The main challenge in this chapter is in combining the "upstairs" method to represent minus with the corner piece. Make sure your students are able to recognize the dimensions of the uncovered rectangle, as most of the work in this chapter depends on it.

Lesson Notes

- **Lesson 1,** Multiplying Monomials, page 56: The ideas presented in this lesson are the foundation of all remaining lessons in this chapter.
- **Lesson 2,** The Uncovered Rectangle, page 57: Though there are probably other ways to show multiplication with minus signs, it is best to follow the format given in this lesson, as it is the standard that will be followed throughout the program.
- **Lesson 3,** Multiplying Polynomials, page 59: Make sure the students cancel matching upstairs and downstairs blocks to get the final answer.
- **Lesson 4,** The Distributive Law, page 60: Problems 9 and 10 force the student to articulate the distributive law.
- **Lesson 5,** More Multiplying, page 61: Make sure the students appreciate that the final, uncovered part of the answer is a rectangle, as it should be, and that it has the right dimensions. Make sure they understand what is meant by the "clockwise" end of the blocks.
- **Lesson 6,** Division and Fractions, page 63: If students have trouble following the examples, start by working out the corresponding multiplication on the overhead projector, then work back.
- **Lesson 7,** More Dividing, page 65: Some trial and error is necessary here.
- **Lesson 8,** Squares and Square Roots, page 67: Students are not asked to find the "square root" of expressions such as $4x^2$, since dealing with the cases of $x > 0$ and $x < 0$ would require more sophistication than beginners usually have. And not dealing with the cases would be quite misleading.
- **Lesson 9,** Three Factors, page 68: The work with three factors is limited by the absence of three-dimensional Lab Gear blocks to represent such quantities as x^3 or x^2y. However, building such blocks out of construction paper should help give the concept more substance. This introduction plants the seed for more work in future years.

Exploration 1 Make a Rectangle

Take blocks for each expression. Arrange the blocks into a rectangle. Write a multiplication equation relating the length, width, and area of the rectangle.

1. $x^2 + 7x$
2. $x^2 + 7x + 6$
3. $x^2 + 7x + 10$
4. $x^2 + 7x + 12$
5. $x^2 + 8x + 12$
6. $x^2 + 13x + 12$

Exploration 2 Make a Square

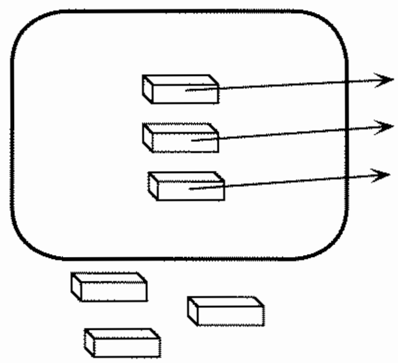
For each problem, arrange the blocks into a square. Not all are possible. Write an equation relating the side length and area of the square.

1. 36
2. 49
3. 40
4. $4x^2$
5. $9x^2$
6. $x^2 + 2x + 1$
7. $x^2 + 6x + 9$
8. $4x^2 + 4x + 1$
9. $x^2 + 8x + 4$
10. $x^2 + 4x + 16$
11. $9x^2 + 12x + 4$
12. $x^2 + 2xy + y^2$

Multiplying Monomials

In some cases, the methods you learned for integer multiplication with the Lab Gear can be used with variables.

For example, to simplify $-3\,(-x)$, start with zero. Then *take off* $(-x)$ three times. What is left is $3x$, as shown by this figure.



For each expression, write an equivalent expression without parentheses.

1. $(-2)\,x$
2. $-3\,(-y^2)$
3. $2\,(-xy)$
4. Look at problems 1, 2, and 3, and complete the following statements:
   a. When multiplying two monomials that are both preceded by a minus sign, the product __________.
   b. When multiplying a monomial that is preceded by a minus sign times a monomial that is not, the product __________.

Exploration 3 Positive or Negative?

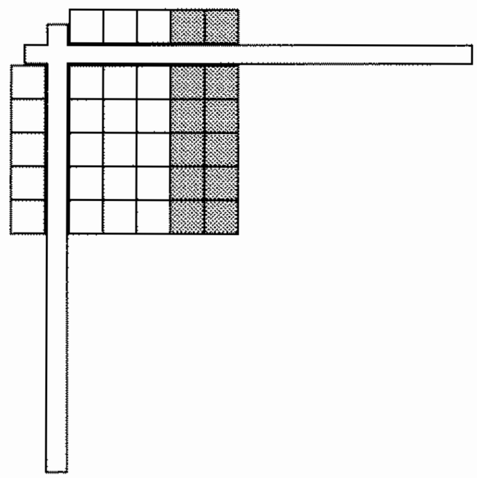
For each expression, write P, N or 0, depending on whether it is positive, negative, or zero. (Try various values for the variables to help you decide. For example, –2, 0, and 2.) Explain your answers.

1. $5x$
2. $-2x^2$
3. $-9y$
4. $5y^2$

The Uncovered Rectangle

This lesson will help prepare you to use the corner piece when minus signs are involved.

For example, this figure shows the multiplication 5(5 – 2).

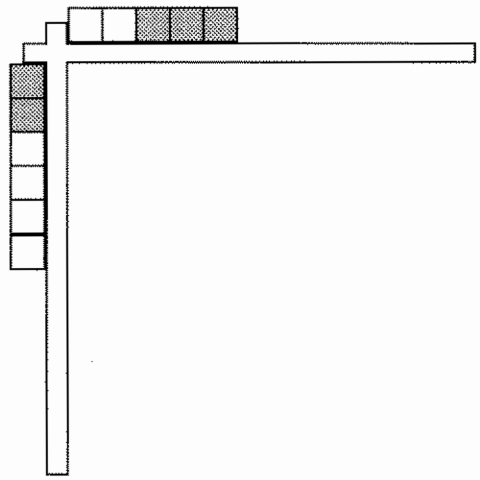


Remember that the shaded blocks are upstairs. Look at the rectangle of downstairs blocks that are not covered by upstairs blocks. We call this the *uncovered rectangle.* The answer to the multiplication is represented by this uncovered rectangle with dimensions 5 and 5 – 2. Of course, the product is 5 times 3, or 15, which is the answer you get when you cancel upstairs and downstairs blocks.

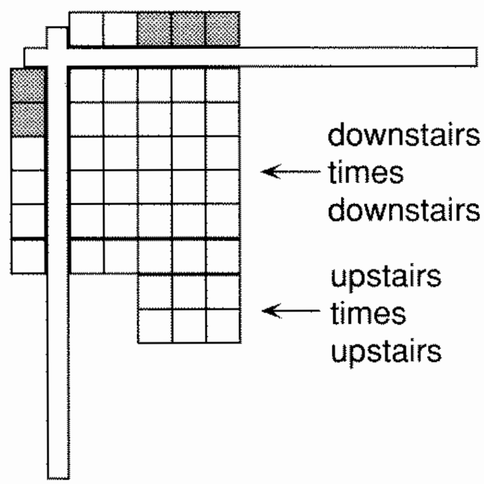
1. Use the corner piece to show the product 3(7 – 2), and sketch your solution.

When there are minus signs in both factors, as in the case of (6 – 2)(5 – 3) you need to set up the problem as shown.

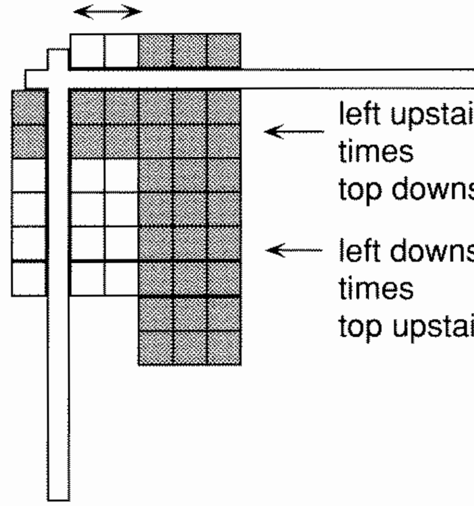
- Put the upstairs blocks at the top end of the left blocks, and the right end of the top blocks. (You can think of it as the *clockwise* end of the blocks.)



- First, multiply the downstairs blocks only.
- Then multiply the upstairs blocks by each other. Since –2(–3) = 6, these blocks must appear downstairs somewhere. You will soon find out why it is convenient to place them at the bottom right as shown on the figure.



- Finally, multiply upstairs blocks on the left with downstairs blocks above, and vice versa, placing them as shown.

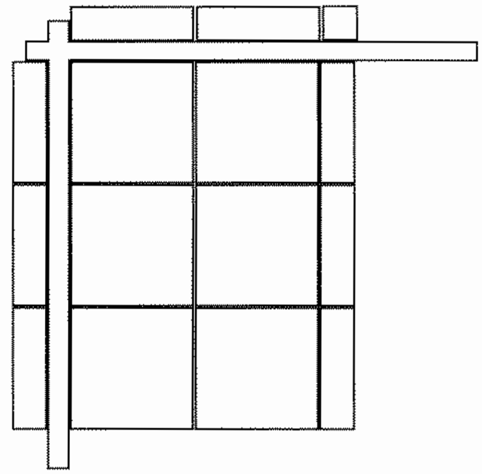


You can now see that the answer (4 times 2 = 8) is shown by the uncovered rectangle.

2. Use the corner piece to show the product (5 – 2)(7 – 4). Sketch your solution.
3. Using arithmetic, the short way to find the answer to the above problem is to multiply 3 times 3. The long way is to multiply every number by every number: 35 – 14 – 20 + 8. Explain how both of these ways can be seen in the blocks.

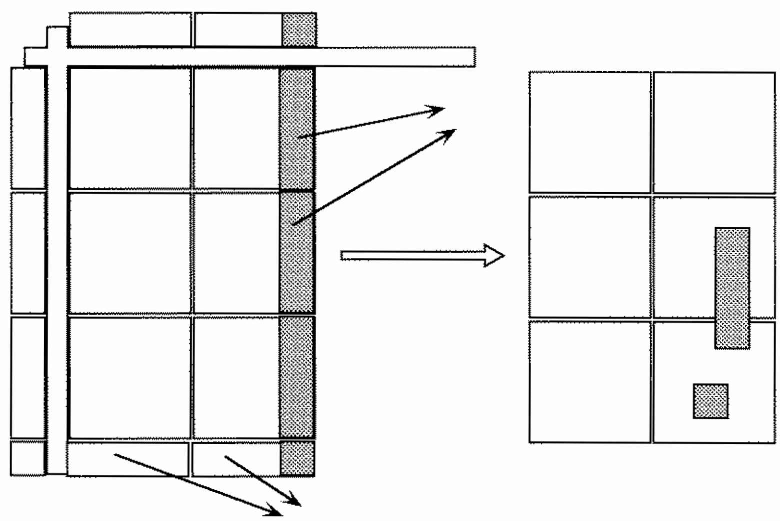
Multiplying Polynomials

This figure will help you remember how to use the corner piece to multiply algebraic expressions. Notice that *every block across the top is multiplied by every block along the side.*



1. Write the multiplication shown by this figure.

This figure shows how to use the corner piece to find the product of polynomials when minus signs are involved. The multiplication is $(3x + 1)(2x - 1)$. Remember that the shaded blocks are *upstairs.*



2. Write the product for this multiplication. Explain what was done to the blocks after using the corner piece.

Notice that, inside the corner piece, the uncovered rectangle has dimensions $3x + 1$ and $2x - 1$. These are the original polynomials we multiplied.

Use the Lab Gear to find these products. Sketch the process as in the above example for problems 3, 6, 9, and 12.

3. $2x(x - 1)$
4. $y(y + 4)$
5. $3x(x + y - 5)$
6. $2y(2x - y + 6)$
7. $3x(2x - 3)$
8. $(x + 5)(3x - 2)$
9. $(y - 4)(y + 3)$
10. $(2x + 4)(x + y + 2)$
11. $x(2y - 5 + x)$
12. $(x - 2)(2x + 3)$
13. $(y + 5)(y + 2x - 3)$
14. $(2y - x - 3)(y + x)$

☑ Self-check

5. $3x^2 + 3xy - 15x$
8. $3x^2 + 13x - 10$
13. $y^2 + 2xy + 2y + 10x - 15$

The Distributive Law

You have investigated ways to remove parentheses from expressions when the parentheses are preceded by a plus or a minus sign. Now look at the question of removing parentheses from expressions involving multiplication.

Use the corner piece to find these products.

1. $x(5 + y)$
2. $(5 + y)x$
3. $y(x + 5)$
4. $(x + 5)y$
5. $5(x + y)$
6. $x(y - 5)$
7. $y(5 - x)$
8. $5(y - x)$
9. Explain how you can correctly remove parentheses from an algebraic expression, when they are preceded or followed by a multiplication.

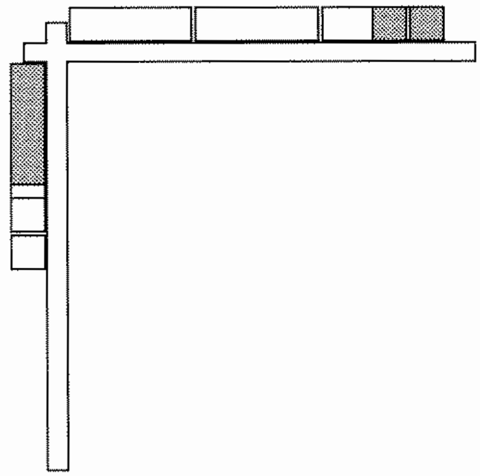
Write equivalent expressions without the parentheses.

10. a. $r(s + t)$
    b. $(r - s)t$
11. a. $-1(x + y)$
    b. $-1(-x + y)$
    c. $-1(x - y)$
12. a. $-(x + y)$
    b. $-(-x + y)$
    c. $-(x - y)$

More Multiplying

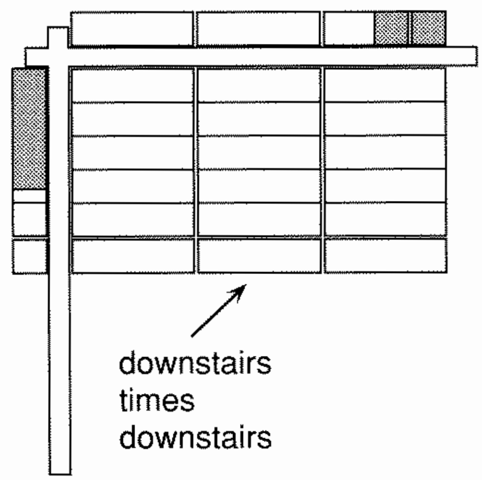
Look at this way to use the Lab Gear to model the multiplication of polynomials when there are minus signs in both factors.

1. Write the polynomials being multiplied.

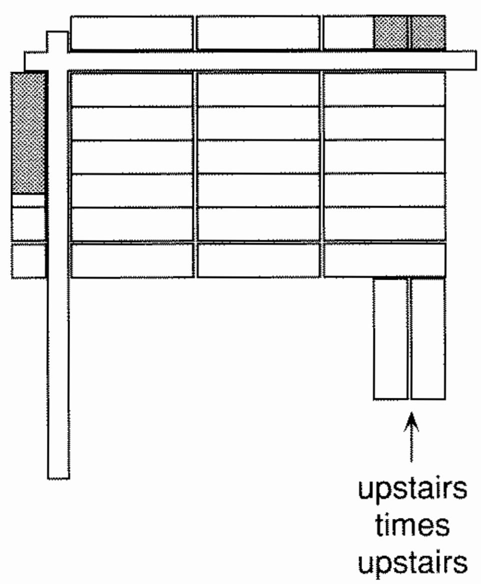


The method you will follow is to multiply all the blocks down the left side by all the blocks across the top. Always be sure to position the upstairs blocks as shown in the figure—they go on the *clockwise* ends of the downstairs blocks.

- First multiply the downstairs blocks only.



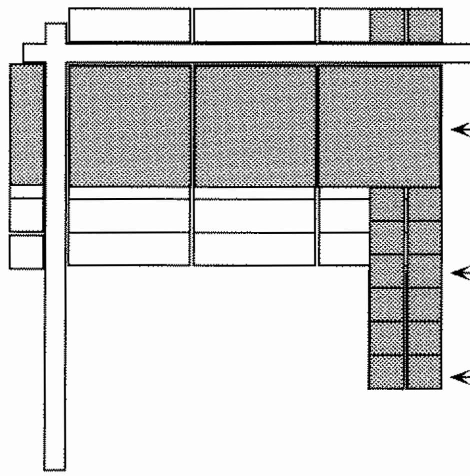
- Then, multiply the upstairs blocks. Since $(-x)(-2) = 2x$, you must put that downstairs somewhere. It is convenient to place the blocks *directly below the –2*.



Now, multiply the upstairs blocks on the left, by the downstairs blocks across the top, and vice versa.

- Multiply left upstairs times top downstairs.
- Now, multiply left downstairs times top upstairs.

2. Write the dimensions of the uncovered rectangle.
3. Write the answer to the multiplication.



For each problem, use the Lab Gear to multiply. Make the uncovered rectangle, then cancel any matching blocks. Write the product.

4. $(4 - x)(2x - 3)$
5. $(y - 5)(2y - 1)$
6. $(3x + 1)(2 + x)$
7. $(6 + y)(2y + 4)$
8. $(3x + 1)(x - 2)$
9. $(y - 4)(2y + 2)$
10. $(2x - 3)(5 - x)$
11. $(y - 3)(y - 5)$
12. $(5 + x)(3x - 3)$
13. $(6 - x)(2x - 3)$
14. $(2y - 2)(6 - y)$
15. $(3x - 1)(2 + x)$

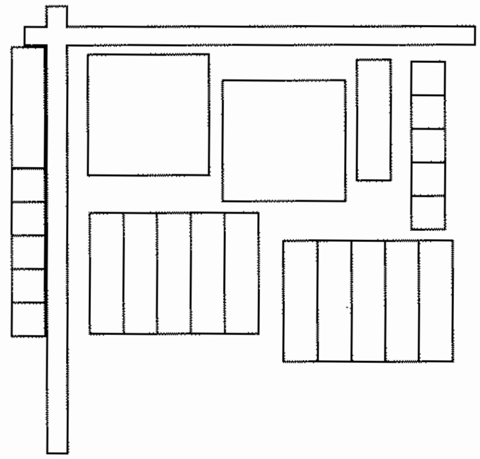
☑ Self-check

4. $-2x^2 + 11x - 12$
10. $-2x^2 + 13x - 15$

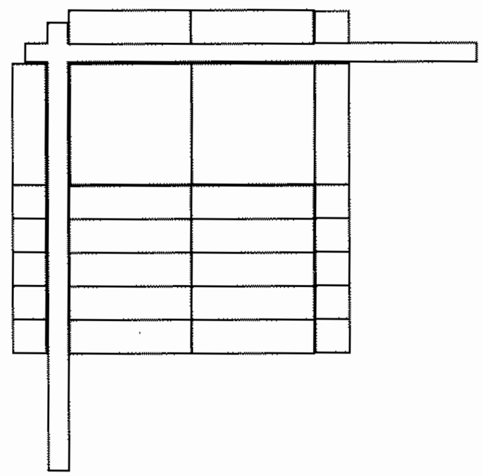
Division and Fractions

This figure shows the division, $\frac{(2x^2 + 11x + 5)}{(x + 5)}$.

- The numerator is inside the corner piece, the denominator is to the left.



- You arrange the numerator blocks to get a rectangle with one side matching the denominator. The other side is the quotient, and is shown across the top of the corner piece.

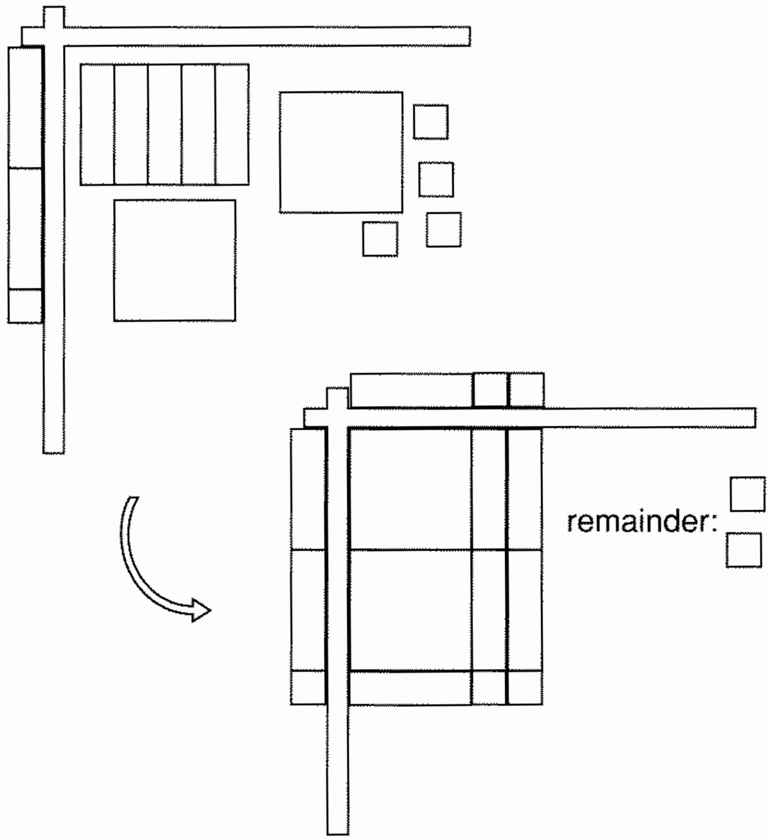


1. Write the quotient.

The denominator was a factor of the numerator, and a rectangle was formed with no pieces left over.

However, in some cases there will be a remainder.

For example, look at these figures.



2. Write the division shown.
3. Write the quotient.
4. Write the remainder.

Use the Lab Gear to show these divisions. Write each quotient and remainder.

5. $\dfrac{6x+4}{2}$
6. $\dfrac{2x^2+4x}{2x}$
7. $\dfrac{9x+3}{3}$
8. $\dfrac{2x^2+6x+4}{x+2}$
9. $\dfrac{3x^2+10x+5}{x+3}$
10. $\dfrac{x^2+3x+8}{x+2}$
11. $\dfrac{4x^2+10x+10}{2x+5}$
12. $\dfrac{2x^2+5x+7}{2x+3}$
13. $\dfrac{2x^2+7x+3}{x+3}$

☑ Self-check

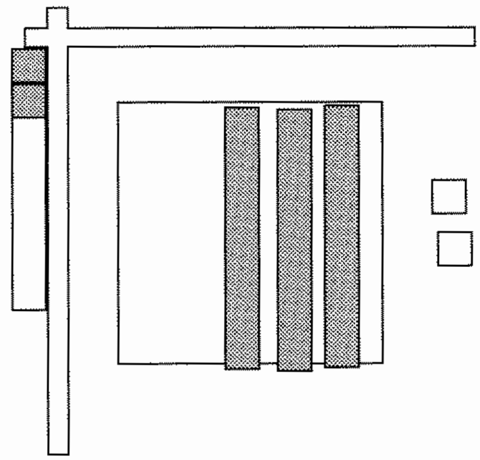
8. $2x+2$

12. $x+1$, r 4

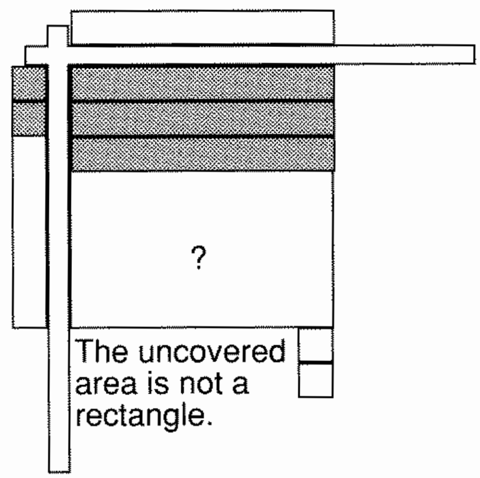
More Dividing

If there is a minus involved, division problems are much harder. Look at this figure.

1. Write the division shown.

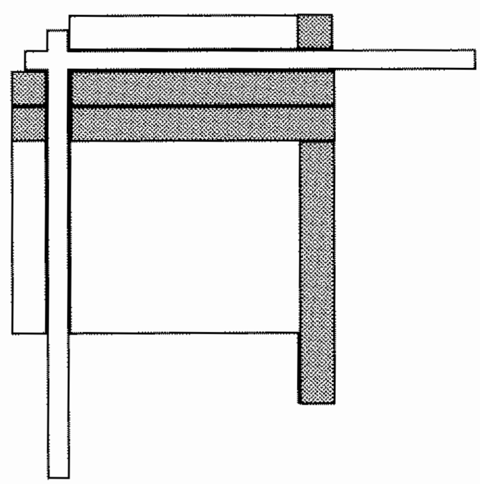


- The arrangement shown in the first try is not correct since the uncovered area is not a rectangle. However, the y^2 and the $-2y$ are placed correctly, because they match the $y - 2$ on the left.



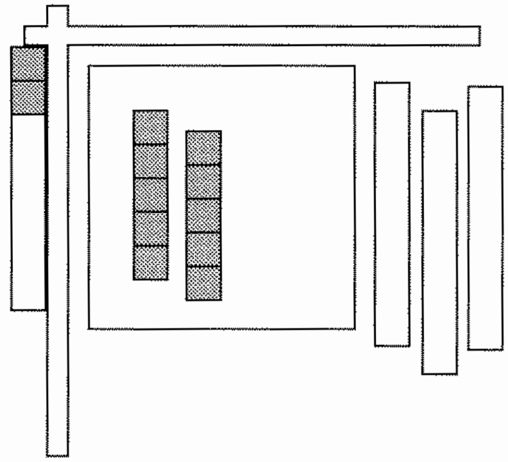
- Moving the other $-y$ does the job. The uncovered area is now a rectangle.

2. Write the quotient.

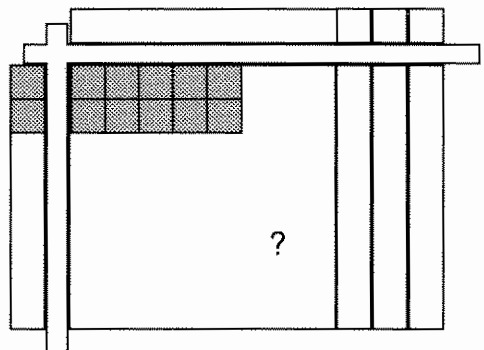


The next example shows once again that adding zero is a useful technique.

3. Write the division shown.



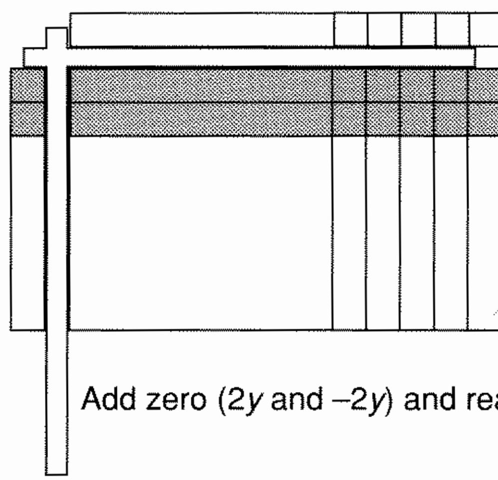
Here is an attempt at arranging the quantity inside the corner piece into a rectangle. This did not work.



The unshaded area is not a rectangle.

However, after adding zero in the form of $2y$ upstairs and $2y$ downstairs, it is possible to get an uncovered rectangle.

4. Write the quotient.



Add zero ($2y$ and $-2y$) and rearrange.

Try these division problems with the Lab Gear.

5. $\dfrac{y^2 + y - 6}{y + 3}$

6. $\dfrac{3x^2 - 9x + 6}{x - 2}$

7. $\dfrac{2x^2 + 7x + 3}{2x + 1}$

8. $\dfrac{y^2 - 6y + 8}{y - 4}$

9. $\dfrac{x^2 + 5x + 1}{x - 1}$

10. $\dfrac{3x^2 + 13x + 4}{x + 4}$

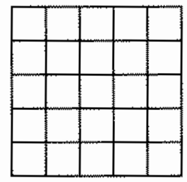
11. $\dfrac{x^2 - x - 6}{x + 2}$

12. $\dfrac{2y^2 + y + 5}{y - 2}$

Squares and Square Roots

The *square* of a positive number can be shown by the area of a square having that number as a side. The positive *square root* of a number can be shown by the side of a square having that number as its area.

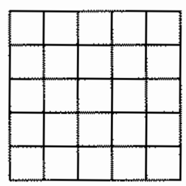
- For example, to find the square of 5, make a square with side 5, and find its area. The answer is 25. You write $5^2 = 25$.
- To find the positive square root of 25, make a square with area 25, and find its side. The answer is 5. You write $\sqrt{25} = 5$.



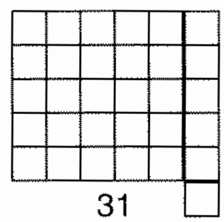
The square root symbol is called a *radical.*

1. Find the square of 9.
2. Find the square root of 9.
3. Find the square root of 36.

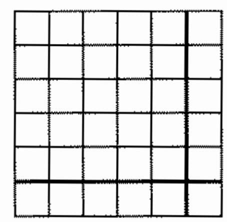
If a number cannot be made into a square with the blocks, its square root is not a whole number. For example, 31 is greater than 25, and smaller than 36, and cannot be shown by a square. Its square root is greater than 5 and less than 6. A calculator provides a more exact value, 5.567764363.



25



31



36

For each number below, the positive square root is between two consecutive whole numbers. Write those two numbers.

4. 42
5. 87
6. 52
7. 114

Find the squares of these expressions. Use the corner piece.

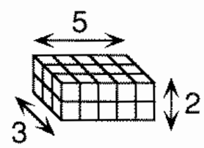
8. $2x$
9. $2 + x$
10. $3y$
11. $3 + y$

☑ Self-check

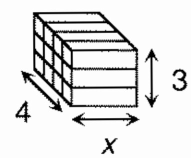
8. $4x^2$

Three Factors

The multiplication equation $3 \cdot 5 \cdot 2 = 30$ can be shown with the Lab Gear like this.



The multiplication expression $3 \cdot 4 \cdot x$ looks like this.



Build these multiplication expressions with your blocks.

1. $5 \cdot 2 \cdot y$
2. $4 \cdot x \cdot 2$
3. $3 \cdot y \cdot y$
4. $8 \cdot 1 \cdot x$
5. $x \cdot y \cdot 5$
6. $x \cdot 2 \cdot x$

Using heavy paper, make boxes to represent each of these expressions.

7. $x^2y = x \cdot x \cdot y$
8. $xy^2 = x \cdot y \cdot y$
9. $x^3 = x \cdot x \cdot x$
10. $y^3 = y \cdot y \cdot y$

When a number is multiplied by itself three times, as in problems 9 and 10, it is called **cubing** the number. For example, 8 is the cube of 2, since $8 = 2 \cdot 2 \cdot 2 = 2^3$.

Notice that when you are working with three factors, upstairs no longer means minus!

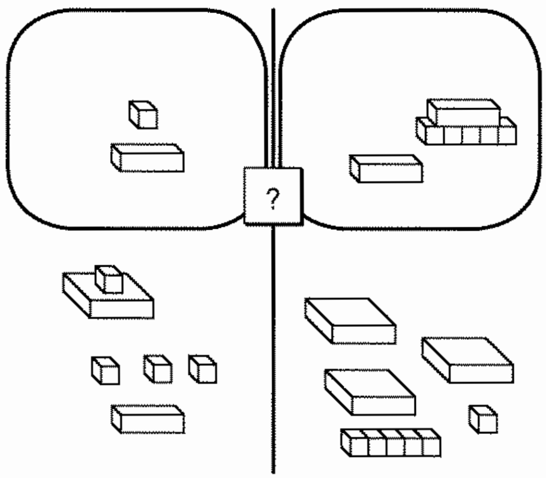
11. Why do you think x^3 is called the cube of x?
12. What do you think "cube root" means? What number would be the cube root of 27?

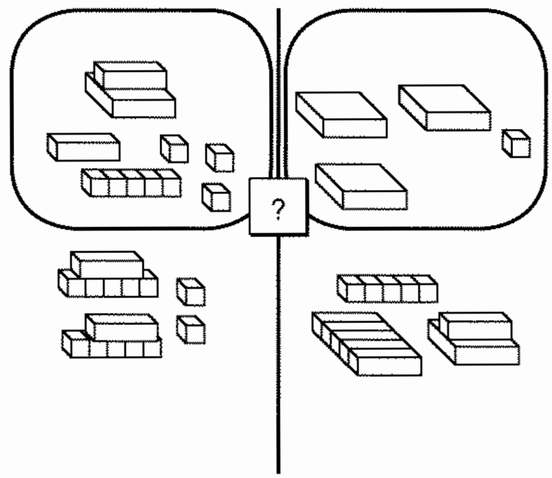
Exploration 4 Which is Greater?

For each of these problems, compare the two expressions by trying different values for x. Good values to try are -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, and 2. Rewrite each sentence, substituting the correct symbol ($<$, $\leq$, $=$, $\geq$, $>$) for the question mark. ($\leq$ means **less than or equal to**, and $\geq$ means **greater than or equal to**.) Explain your thinking if you say it is impossible to decide which is greater.

1. x **?** $2x$
2. x^2 **?** x
3. x^2 **?** -1
4. x^2 **?** 0

For these problems, simplify both sides first. Use the Lab Gear.

5. 

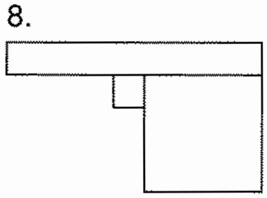
6. 

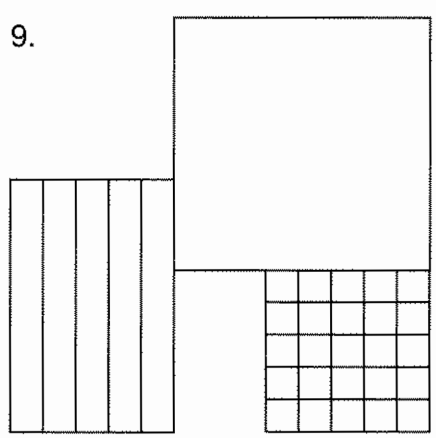
Exploration 5 Perimeter

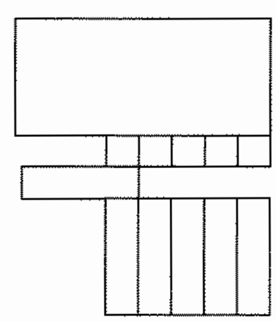
Use an xy-block and a 5-block to make shapes with these perimeters. Sketch the shape in each case.

1. $2x + 2y + 2$
2. $2x + 2y + 10$
3. $2y + 12$
4. Repeat problems 1–3, using a y-block and a $5x$-block.
5. Use another combination of blocks to get a perimeter of $2x + 2y + 2$.
6. Use another combination of blocks to get a perimeter of $2x + 2y + 10$.
7. Use another combination of blocks to get a perimeter of $2y + 12$.

Write the perimeter of each of these figures.

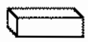
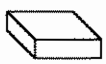
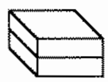
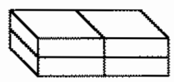
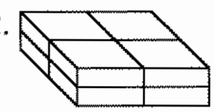
8. 

9. 

10. 

Exploration 6 Volume and Surface Area

Write the volume and surface area for each of these Lab Gear buildings.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 

Describe and sketch the next building in these sequences. Compare your answers with your classmates'. For each, find the volume and surface area.

13. Buildings 1-4.
14. Buildings 5-8.
15. Buildings 9-12.

Exploration 7 Always, Sometimes, or Never?

Write A, S, or N, depending on whether the statement is always, sometimes, or never true. Explain your answers.

1. $3\,(2y-5)=3\cdot 2y-5$
2. $4x\,(1+7)=4x\cdot 1+7$
3. $-3-(y^2+2)=-3-y^2-2$
4. $2x\,(x+5)=2x^2+10x$
5. $10+(3y+9)=10+3y+9$
6. $25-(5-x)=25-5-x$

Chapter 6 Combining Operations

This chapter is a transition into the more advanced uses of the Lab Gear. It extends, concludes, reviews, and combines the concepts and skills that were introduced in previous chapters: parentheses, the distributive law, rectangles, inequalities, perimeter, and surface area.

New Words and Concepts

This chapter introduces the following concepts:

Order of operations
Work with the three familiar **identities**
Long division of polynomials

The first two of these are extremely important. The third topic (long division) is rather complicated, and you may decide that it is not worth the trouble, especially since students will not have a use for polynomial division until second year algebra or even precalculus. The Lab Gear long division algorithm does not parallel every step of the traditional algorithm, but it is an interesting and clever application of many of the techniques that have been learned so far. It requires using both the corner piece and the workmat.

Teaching Tips

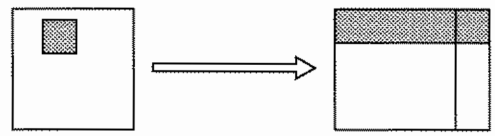
Lessons 1 through 4 pull together the knowledge that has been accumulated in the first five chapters, and prepares the students to put it all to use in the final three chapters. Take the time to make sure the students can verbalize the key ideas. This would be a good time for a "Lab Gear midterm."

Lesson Notes

- **Lesson 1,** Order of Operations, page 72: These exercises do not involve the blocks. They are challenging, and students should be encouraged to work together on them.
- **Lesson 2,** The Distributive Law, page 74: This is the first formal statement of the distributive rule. All the work that has been done until now should have laid the basis for a solid understanding of the law.
- **Lesson 3,** Parentheses Review, page 75: This lesson should help the students consolidate their understanding of parentheses, and it should help you evaluate whether they have mastered the concepts and techniques.
- **Lesson 4,** Three Identities, page 76: Because the students have seen many "Make a Rectangle" and "Make a Square" problems, this lesson should present no conceptual difficulty. See it as an opportunity to start learning to recognize the identities, which will be useful in further work in algebra—factoring, solving equations, completing the square, rationalizing denominators, and so on.
- **Lesson 5,** Long Division With the Lab Gear, page 77 and **Lesson 6**, Long Division Without the Lab Gear, page 80: These lessons are optional.

Exploration 1 Make a Rectangle

As you can see in this figure, it is possible to make an uncovered rectangle from the blocks representing $x^2 - 1$ by adding zero in the form of x and $-x$.



1. Write an equation relating length, width, and area of the uncovered rectangle.

Use the same method to make an uncovered rectangle for these sets of blocks. In each case, write an equation.

2. $y^2 - 1$
3. $25 - x^2$
4. $y^2 - 25$
5. $y^2 - x^2$
6. $y^2 - 9$
7. $4x^2 - 4$

Lesson 1

Order of Operations

Look at the expression, $1 + 2\ (3 + 4)$. Mathematicians agree that it is equal to 15.

1. In what order would you do the calculations to get 15?
2. The expression is not equal to 21, but explain why someone might think it is.
3. The expression is not equal to 11, but explain why someone might think it is.

To prevent the same expression from having several different meanings, mathematicians have agreed on a set of rules to read complicated expressions. This set of rules is called the **order of operations**. It is used not only in mathematics, but also in computer programming languages and in scientific calculators. The rules require that you perform operations in a specific order.

- Exponents and roots first
- Multiplications and divisions next
- Additions and subtractions last

Aside from that, proceed from left to right, and pay attention to grouping symbols such as parentheses. Parentheses are used to override the order of operations rules.

Look at this example. $1 + 2 \cdot 3 + 4 = 1 + 6 + 4 = 11$
This equation is true because multiplications are performed before additions.

However, the addition represented by the second plus sign would be performed before anything else if the operation 3 + 4 was enclosed in parentheses.

$$1 + 2 \cdot (3 + 4) = 1 + 2 \cdot 7 = 15$$

If you have parentheses within parentheses, work from the inside out.

Radical signs and fraction bars also create groups. Study this example.

$\dfrac{\sqrt{1+8\cdot 10}+9}{1-(2+3)}-4=$	work within the radical and the parentheses
$\dfrac{\sqrt{1+80}+9}{1-5}-4=$	work within the radical and the denominator
$\dfrac{\sqrt{81}+9}{-4}-4=$	calculate the radical
$\dfrac{9+9}{-4}-4=$	calculate the numerator
$\dfrac{18}{-4}-4=$	do the division
$-4.5-4=$	do the subtraction
-8.5	

Keeping in mind the order of operations, insert as many pairs of parentheses as needed to make these equations true.

4. $4\cdot 2+3=20$
5. $\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$
6. $5\cdot 3-2+6=35$
7. $3^2+2\cdot 7-4=33$
8. $\frac{1}{3}\cdot 6+4\cdot\frac{2}{6}-\frac{1}{3}=\frac{7}{3}$
9. $1-2\cdot 2+5\cdot 6=-42$
10. $4+6\cdot 2\cdot 5-3=40$
11. $3+1\cdot 7-2^2\cdot 9-7=24$
12. $2\cdot 8\cdot\frac{1}{4}+\frac{2}{3}\cdot 2-\frac{1}{2}=22$

Insert radical signs and/or parentheses to make these equations true.

13. $\dfrac{5\cdot 12+4}{6-2}=5$
14. $\dfrac{2+6\cdot 3^2}{2}=6$
15. $3\cdot 2+4\cdot 2^3=12$

☑ Self-check

4. $4\cdot(2+3)=20$
10. $(4+6)\cdot 2\cdot(5-3)=40$
14. $\sqrt{\dfrac{(2+6)\cdot 3^2}{2}}=6$

The Distributive Law

1. Use the corner piece to work out this product. $(x + y + 1)(x + 5)$
   - Remember that every block on the left must be multiplied by every block across the top. Sketch your result.
2. Use the corner piece to work out this product. $(x + y - 1)(x + 2)$
   - Remember to cancel matching upstairs and downstairs blocks at the end. Write the product.
3. Use the corner piece to work out this product. $(x + y - 5)(-x + y + 1)$
   - Remember to start by multiplying the downstairs blocks with each other.
   - Then, multiply the upstairs blocks with each other (careful!).
   - Next, multiply upstairs times downstairs and downstairs times upstairs. If you did it correctly, the uncovered area should be a rectangle of the right dimensions.
   - Finally, cancel matching upstairs and downstairs blocks. Write the product.

To do multiplications like the above without the Lab Gear, you have to remember the **distributive law**, which states that you must *multiply each term in the first polynomial by each term in the second.* Then you can simplify by combining like terms.

4. Rework problems 1-3 without the Lab Gear. For each one, make sure you get the same answer as before. If not, find your mistake!

Do these multiplications by algebra first, then check your answers with the Lab Gear.

5. $(2x + 3)(3x - 2)$
6. $(-5 + 2x)(x + y - 3)$
7. $(y - 3)(y + 4)$
8. $(2x + y - 5)(y - x + 4)$
9. $(3x - 4)(x + 5 - y)$
10. $(2y - 1 - x)(2x + y - 3)$
11. $(x + y + 1)(2x + y - 1)$
12. $(2x + y - 5)(2x + y + 1)$
13. $(-x + y + 5)(x + y - 5)$

☑ Self-check

1. $x^2 + xy + 6x + 5y + 5$
2. $x^2 + xy + x + 2y - 2$
3. $-x^2 + y^2 + 6x - 4y - 5$

Parentheses Review

Write a summary about the role of parentheses in algebra. Use sketches of the Lab Gear to explain the various rules that you describe. Be sure to cover these four points.

- Parentheses and order of operations
- Removing parentheses that are preceded by a plus
- Removing parentheses that are preceded by a minus
- Removing parentheses that are preceded or followed by a multiplication sign

Exploration 2 Always, Sometimes, or Never?

Always, sometimes, or never true? is an important question in algebra.

1. Make up several algebra statements that are always true.
2. Make up several algebra statements that are sometimes true.
3. Make up several algebra statements that are never true.
4. Discuss your answers with your classmates.

Exploration 3 Positive or Negative?

For each quantity below, indicate with P, N or 0, if it can be positive, negative, or zero. In problems 4, 5, and 6, try replacing with the values –3, 0, 3.

1. 4^3
2. -4^3
3. $(-4)^3$
4. $2y^3$
5. $-2y^3$
6. $(-2y)^3$
7. Which of the expressions in problems 1, 2, and 3 are equal to each other? Explain.
8. Which of the expressions in problems 4, 5, and 6 are equal to each other? Explain.
9. Compare what you discovered about the signs of cubes to what you know about the signs of squares.

Three Identities

Use the corner piece to find these products.

1. $(x + 3)^2$
2. $(x + 5)^2$
3. $(x + y)^2$
4. Explain how to find the square of a sum without the Lab Gear.
5. True or False: The square of a sum is equal to the sum of the squares. Explain, using a sketch of problem 3.

Use the corner piece to find these products.

6. $(y - 2)(y + 2)$
7. $(y - 5)(y + 5)$
8. $(y - x)(y + x)$
9. What pattern did you discover in problems 6-8?

Use the corner piece to find these products.

10. $(y - 3)^2$
11. $(y - 5)^2$
12. $(y - x)^2$
13. Explain how to find the square of a difference.
14. True or False: The square of a difference is equal to the difference of the squares. Explain, using a sketch of problem 12.

As you know, **identities** are equations that are *always true.* The three that are shown in problems 3, 8, and 12 are especially important and useful. For example, using the identity in problem 12:

$$(2x - 5)^2 = (2x)^2 - 2\,(2x)(5) + 5^2 = 4x^2 - 20x + 25$$

Do these problems by using one of the identities, then check your answers with the Lab Gear.

15. $(2x - 3)^2$
16. $(2y + x)^2$
17. $(4y - 1)(4y + 1)$
18. $(3x + 4)(3x - 4)$
19. $(3y + 5)^2$
20. $(5x - 2y)^2$

Do you think there is a pattern for the square of trinomials? Use the corner piece to experiment with these problems.

21. $(x + y + 2)^2$
22. $(x + y - 5)^2$
23. Describe the pattern you discovered in the two problems above.
24. What is $(a - b + c)^2$ equal to? Use the pattern you discovered, then check your answer by using the distributive rule very carefully.

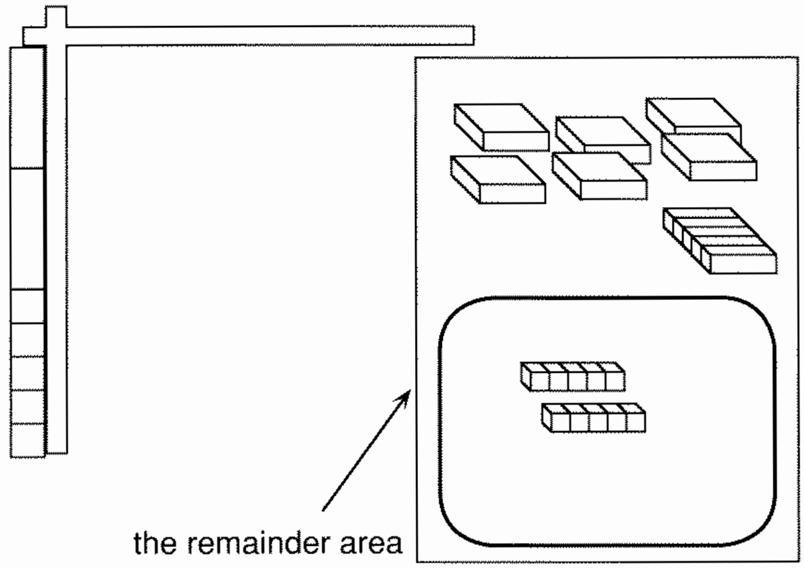
Long Division With the Lab Gear

As you know, it is sometimes possible to divide polynomials with the corner piece. Some divisions are fairly easy to work out by trial and error. Others are harder. Still others cannot be solved with the blocks. The following method will work whenever the division can be modeled with the Lab Gear. It requires one side of the workmat, which will be called the remainder area, as well as the corner piece.

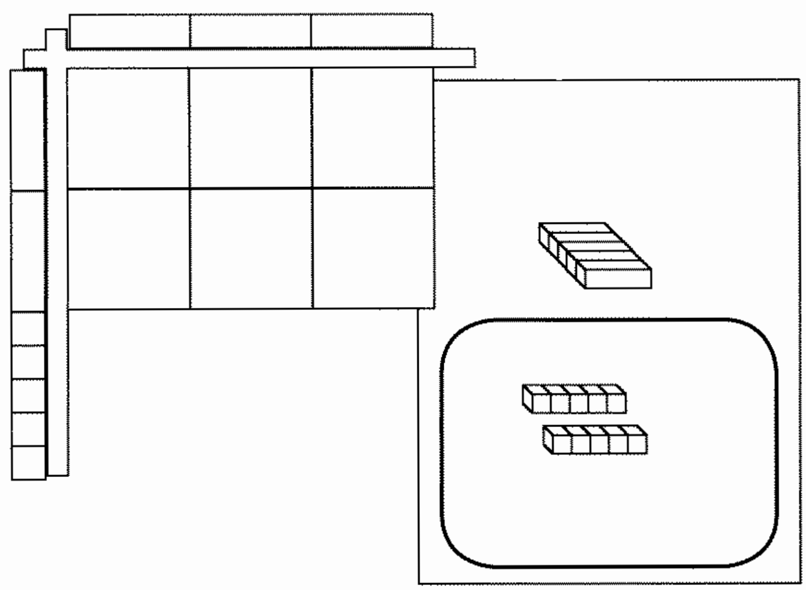
Study this example. Follow along with your blocks.

- As usual, the denominator is placed on the left side of the corner piece. But this time the numerator is placed in the remainder area.

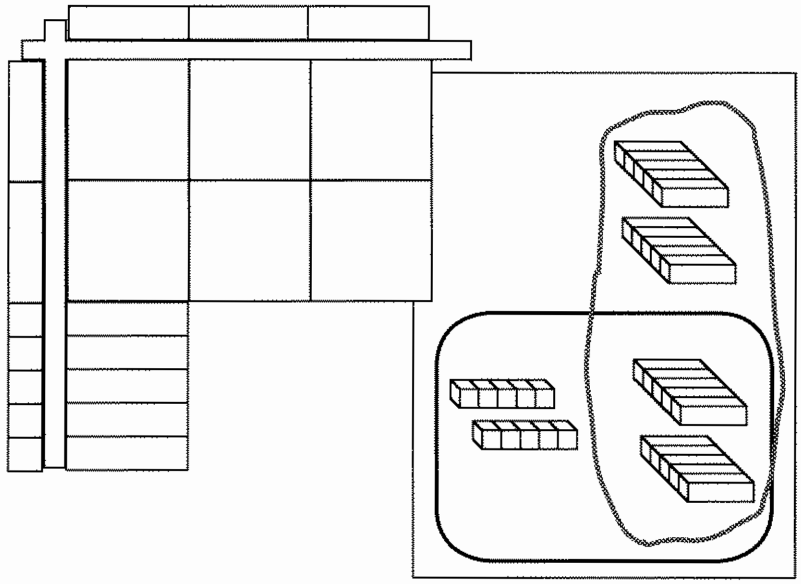
1. Write the division.



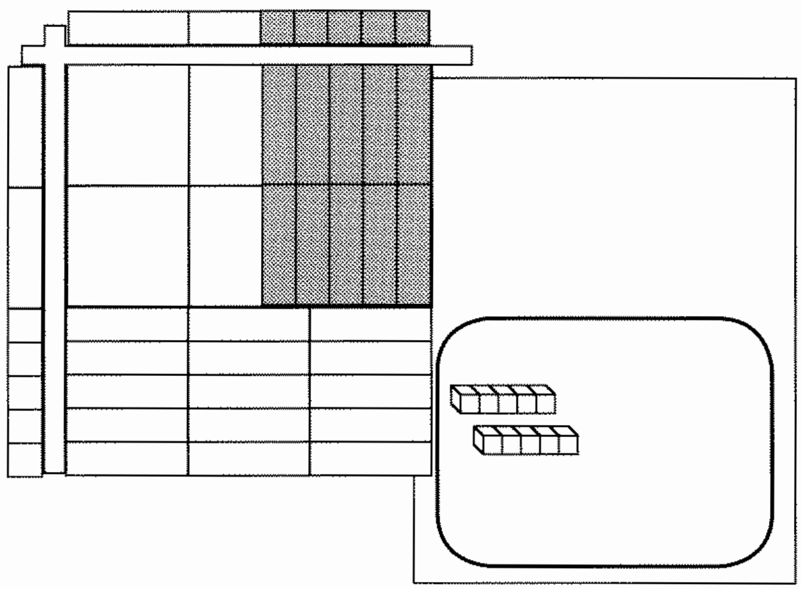
- The first step is to move the x^2-blocks from the remainder area to the inside of the corner piece. Line them up to match the x-blocks on the left. This shows that you need three x-blocks across the top.



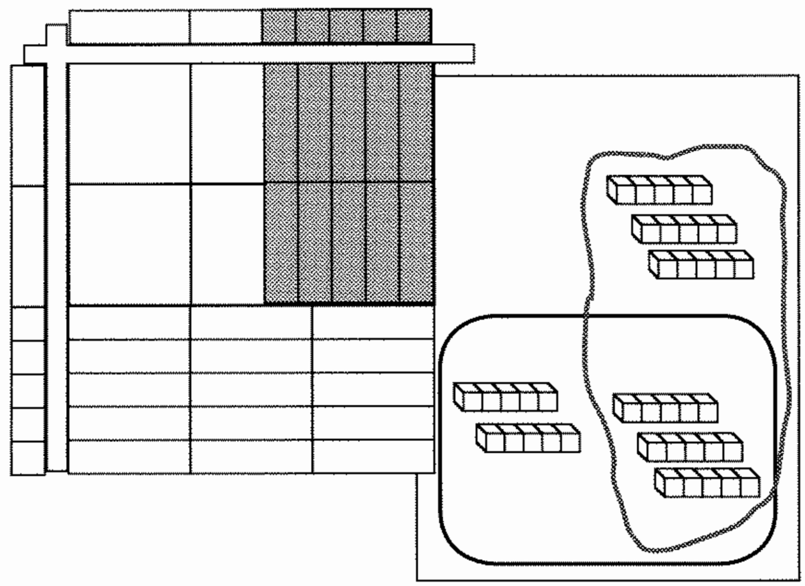
- Now complete the rectangle by moving fifteen x-blocks to the inside of the corner piece. Only five are in the remainder area, so zero must be added in the form of $10x$ and $-10x$.



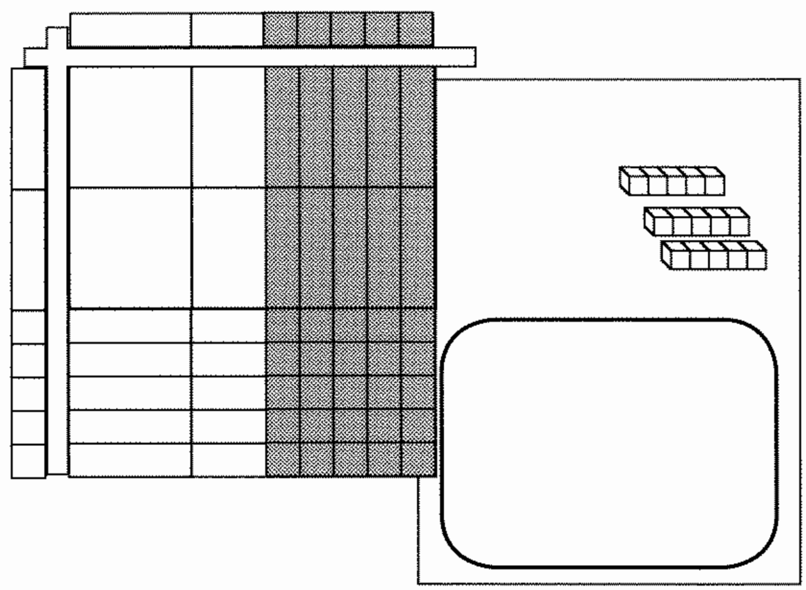
- Be sure to move the x-blocks from the minus area to the upstairs of the inside of the corner piece. The other crucial thing to keep in mind is that they must appear vertically, so that they reveal the rest of the quotient. In this case, -5 must appear across the top.



- Again, complete the rectangle. –25 will do it, but only –10 is available in the remainder area. Therefore it is necessary to add zero in the form of 15 and –15.



2. Write the quotient and the remainder.



Use this method to do these divisions.

3. $\dfrac{2x^2 + 5x + 3}{x + 1}$

4. $\dfrac{4x^2 - 8x + 5}{2x - 4}$

5. $\dfrac{y^2 + 10y - 6}{y + 4}$

6. $\dfrac{6x^2 - 10x + 13}{3x - 5}$

☑ Self-check

4. $2x$, r 5

Long Division Without the Lab Gear

- To divide polynomials, you set up the problem like you would set up a **long division** with numbers.

$$2x+5\overline{)6x^2+\ 5x-10}$$

- First, divide. To figure out the quotient, start with the question, "How many times $2x$ to get $6x^2$?" and write the answer ($3x$) above the line.

$$\begin{array}{r} 3x \\ 2x+5\overline{)6x^2+\ 5x-10} \end{array}$$

- Then multiply $3x$ by $(2x + 5)$, and write the answer lined up correctly beneath the numerator. This quantity represents a complete rectangle.

$$\begin{array}{r} 3x \\ 2x+5\overline{)6x^2+\ 5x-10} \\ -(6x^2+15x) \end{array}$$

- To move it *out of the remainder*, subtract it from the numerator.

$$\begin{array}{r} 3x \\ 2x+5\overline{)6x^2+\ 5x-10} \\ \underline{-(6x^2+15x)} \\ -10x-10 \end{array}$$

- Now repeat the whole process. Divide $2x$ into $-10x$ to find the quotient; multiply to get a complete rectangle; subtract to move it out of the remainder.

$$\begin{array}{rl} 3x-\ 5 & \longleftarrow \text{divide} \\ 2x+5\overline{)6x^2+\ 5x-10} & \\ \underline{-(6x^2+15x)} & \\ -10x-10 & \\ \underline{-(-10x-25)} & \longleftarrow \text{multiply} \\ 15 & \longleftarrow \text{subtract} \end{array}$$

As expected, this method gives the same answer as the block method. The quotient is $3x - 5$, and the remainder is 15.

Solve these problems. Do each one first with the Lab Gear, then with the method shown above. Compare the way the blocks look to the written way of solving the problem.

1. $\dfrac{2y^2+6y+5}{y+3}$
2. $\dfrac{4x^2-7x+8}{x-1}$
3. Do problems 3-6 from lesson 5 again without blocks. Check to see if you get the same answer.

The following problems cannot be done with the Lab Gear. Use the above method.

4. $\dfrac{x^3+2x^2-5x+9}{x+4}$
5. $\dfrac{4x^3+8x^2-5x-9}{2x^2+x-4}$

☑ Self-check

2. $4x - 3$, r 5

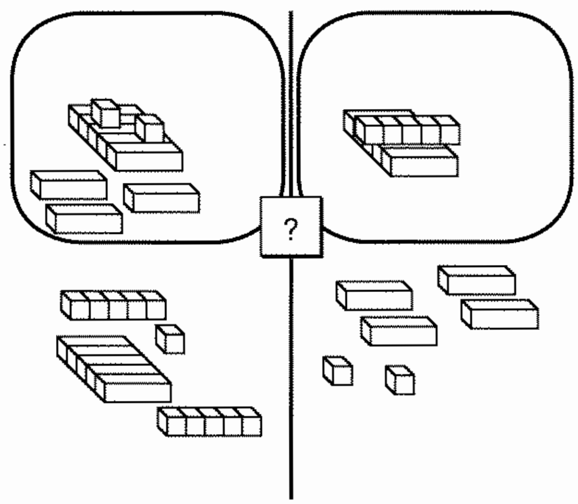
Exploration 4 Which is Greater?

For each of these problems, compare the two expressions by trying various values for x. Good values to try are -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, and 2. Decide, if possible, which symbol ($<$, $\leq$, $=$, $\geq$, $>$) to substitute for the question mark. Explain your answers.

1. x^2 **?** x
2. x^2 **?** $2x^2$
3. x^3 **?** x^2
4. x^3 **?** x

For these problems, write the *Which is Greater?* question that is illustrated by the Lab Gear. Then simplify both sides to answer the question. If it is impossible to tell which is greater, try to say *which values of x make the two sides equal. Which values of x make the left side greater*?

5.



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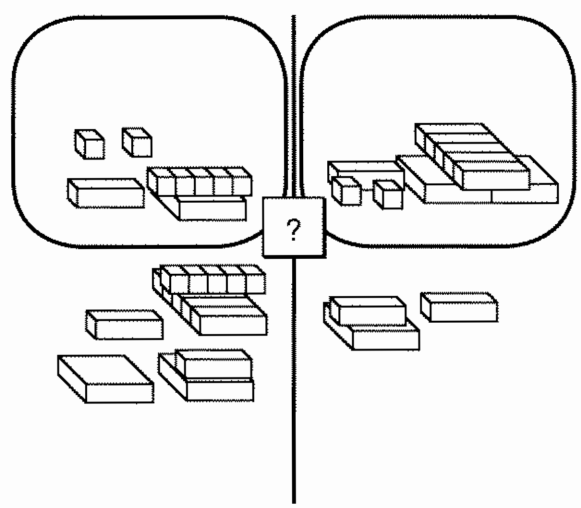


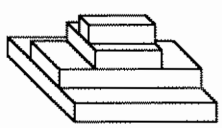
Exploration 5 Perimeter

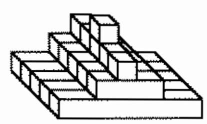
These problems are about the figure formed by the top faces of the blocks.

1. Arrange three blocks so that the perimeter of the resulting figure is $6x + 2y$. How many solutions can you find?
2. Arrange four blocks so that the perimeter of the resulting figure is $8x + 18$. How many solutions can you find?
3. Arrange five blocks so that the perimeter of the resulting figure is $2y + 2x + 12$. How many solutions can you find?

Exploration 6 Surface Area

Find the surface area of these buildings.

1. 

2. 

Chapter 7 Solving Equations

This chapter focuses on simple equation solving.

New Words and Concepts

By now, your students have done a lot of work to lay the groundwork for the **solving of linear equations**. Familiarity with the manipulation of variables, the ability to combine like terms, to simplify expressions, to handle parentheses and minus signs, are all necessary for effective work with equations. Some students may even have developed equation-solving techniques on their own in the context of working on the "Which is Greater?" Explorations. (They get one more chance to do this at the beginning of this chapter.)

The overview of the different types of numbers (**natural**, **integer**, **rational**, **irrational**, **real**, and **complex**) provides a context for equation-solving.

Teaching Tips

Very few equation-solving techniques are given to the student. The "adding zero" trick works, but it is likely that your students will come up with more efficient methods, such as the following:

- Moving quantities from upstairs in the minus area to downstairs outside, and vice versa.
- Moving quantities diagonally (from the minus area on one side to outside the minus area on the other side, or vice versa). This is a shortcut that corresponds to adding or subtracting the same quantity on both sides.
- Moving out all the quantities from both of the minus areas, and moving all quantities from the two outside areas into the minus areas. This corresponds to multiplying both sides by –1.

If some of your students have not discovered these techniques, you can demonstrate them on the overhead projector. Whenever possible, credit a student with the discovery of the tricks, or better yet, ask a student to demonstrate.

After this chapter, you may want to skip directly to the lessons on simultaneous equations in Chapter 9, which follow naturally from the work with linear equations. Doing them later, however, has the advantage of offering a review of this material.

Lesson Notes

- **Lesson 1,** Linear Equations, page 84: Do not give away any techniques yet, except for the ones used in the examples. Encourage students to get help from each other if they are having trouble.
- **Lesson 2,** Solving Equations, page 88: The new element here is the presence of parentheses and the need to use the distributive rule in solving. Students may be tempted to do these problems without the blocks. Insist that they solve at least some of them with the blocks, and compare their work with and without the blocks.
- **Lesson 3,** Solving Techniques, page 89: This is the first explicit mention of equation-solving methods.
- **Lesson 4,** Solving Tricks, page 92: This lesson provides an opportunity for the students who are most proficient with the blocks to share their techniques with the whole class, perhaps on the overhead projector.
- **Lesson 5,** Equations and Numbers, page 94: You can point out to students that their own growth as a math student since kindergarten has been taking them to ever increasing number realms.

Exploration 1 Which is Greater?

For each of these problems, try to decide which of the two expressions is greater. If it is impossible to tell which side is greater, try to tell *which values of x make the two sides equal. Which values make the left side greater?*

Some of the problems may be challenging. If you cannot answer the questions, make a note of the problem, and come back to it later.

1. x **?** $2x + 3$
2. $4x$ **?** $4x + 5$
3. $6x$ **?** $7x^2 + 6x - 7$
4. $8x + 9$ **?** $-x^2 + 8x + 9$
5. $3x^2 + 2x$ **?** $3x^2 + 2x - 5$
6. $3x^2 + 4x$ **?** $3x^2 + 6x + 10$
7. $3x + 4$ **?** $5(x + 6)$

Lesson 1

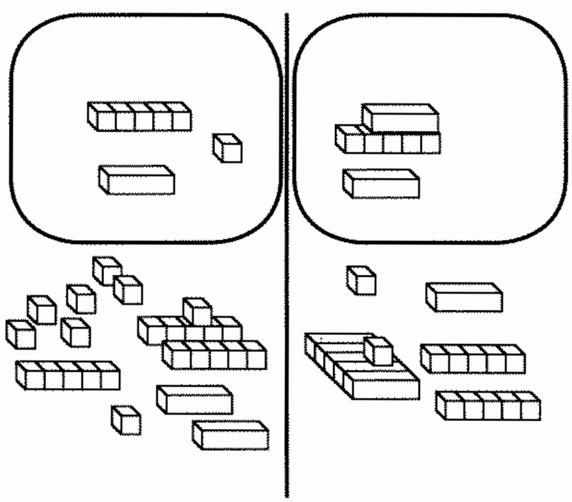
Linear Equations

For each of these statements, indicate always, sometimes, or never true by writing A, S, or N. Explain each answer.

1. $3x = 3x + 5$
2. $3x = 2x + 5$
3. $3x = 2x + x$
4. Write the value of x that makes the statement in problem 2 true. If you did this correctly, you found the **solution** of the equation (you *solved* the equation). Does this equation have other solutions?

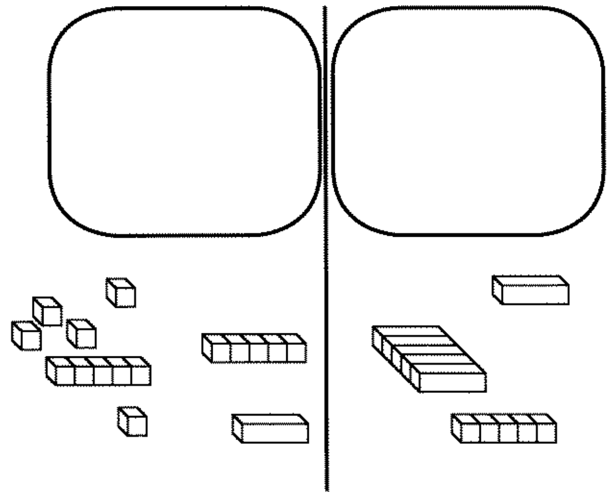
Much of algebra is about solving equations. The easiest equations to solve are the ones where there is only one variable, and it is never raised to a power greater than one. These are called **linear equations in one variable**.

5. Put out blocks to match this figure. Write the equation.

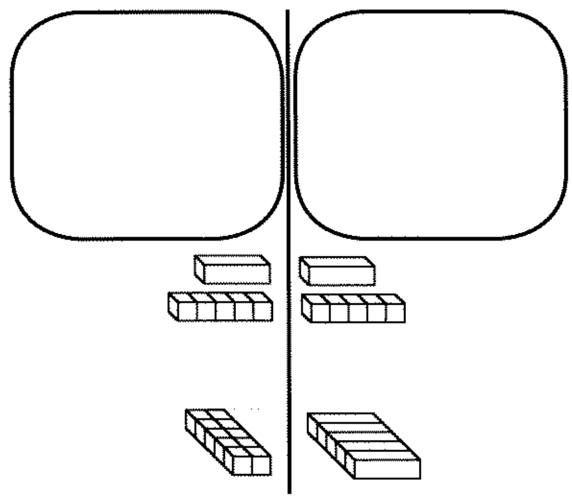


- To solve the equation, start by simplifying each side, by cancelling opposites.

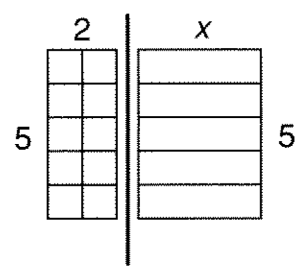
- If you did it correctly, you should have blocks to match this figure.



- Rearrange the blocks like this, matching blocks on the left and right side. Mark the matching blocks in the figure.



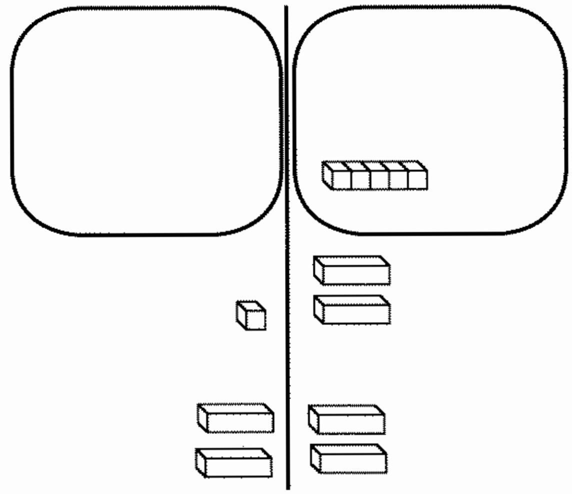
- Look at the other blocks and remember that the two sides are equal. (This is true even though they don't *look* equal. Remember, x can have any value.) You should see the solution to the equation.



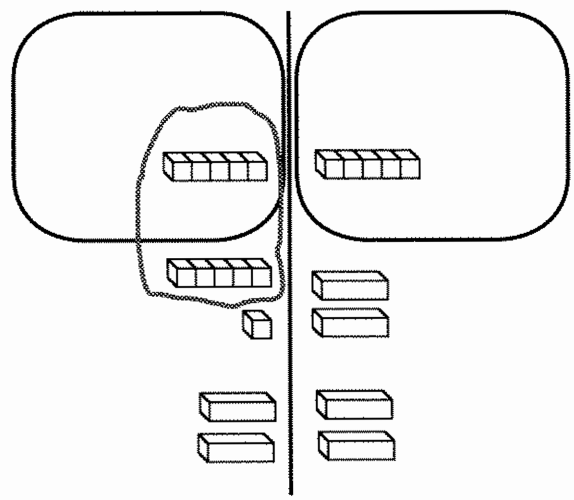
6. Write the solution.

Here is another example. Use the Lab Gear to solve the equation
$2x + 1 = 4x - 5$.

Even with each side fully simplified, and the blocks nicely arranged, it is not easy to tell what the solution is.

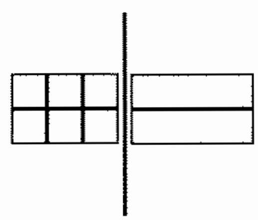


The adding zero trick helps once again.



7. Mark all the matching blocks.

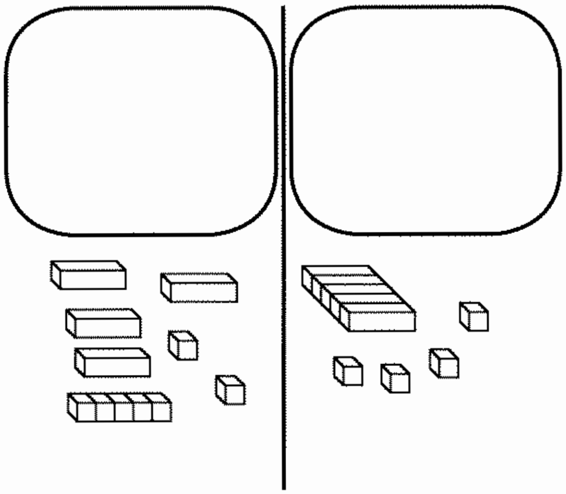
Rearrange the blocks that are left like this.



8. Find the value of x that makes the equation true. Write this solution.

Write and solve these equations.

9.



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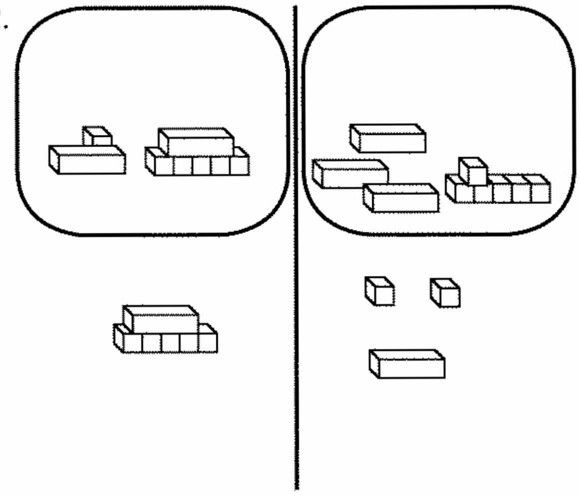

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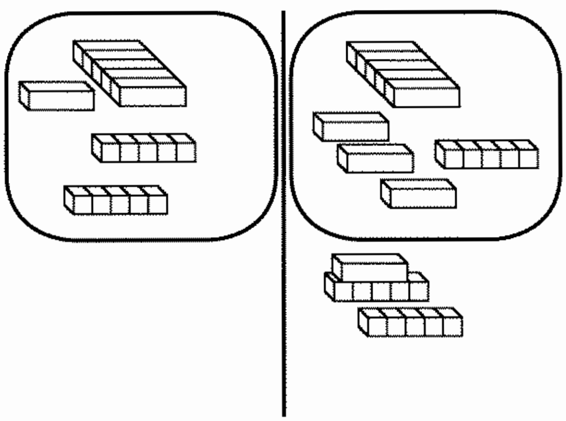

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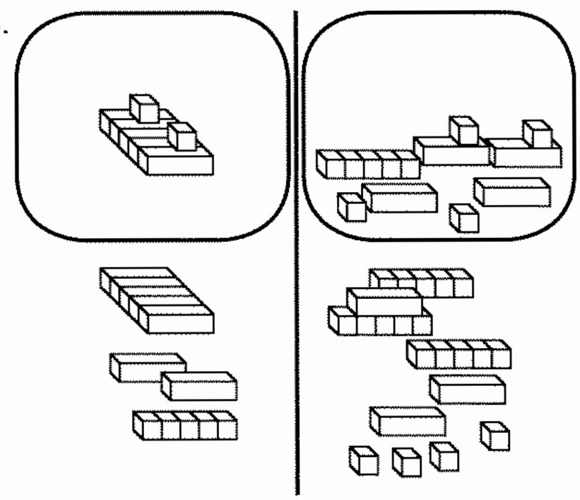

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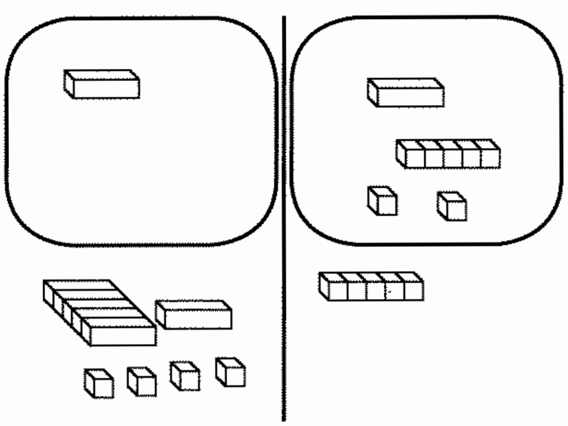


Exploration 2 How Many Solutions?

Think about the number of solutions for each of these equations. Some have one solution, some have two, some have no solutions with rational numbers, and some are identities. (Remember that identities are true for *all* values of the variable.)

Find the solutions. You may use the Lab Gear, but for some of the problems it may not help.

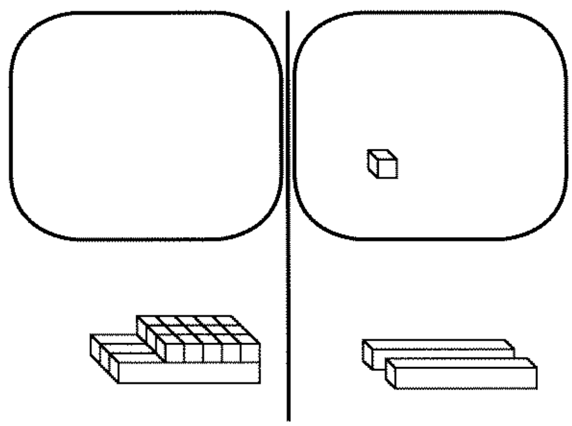
1. $9x + 7 = -2$
2. $2x = 15$
3. $x^2 = 16$
4. $2x + 4x = 6x$
5. $2x + 4x = 8x$
6. $2x \cdot 3x = 6x^2$
7. $2x \cdot 3x = 5x^2$
8. $2x \cdot 3x = 6x$
9. $x^2 = -16$
10. $x^3 = -8$
11. $(x + 2)(x - 3) = 0$
12. $x^2 = 10$

If you had trouble with the last problem, or the last few, discuss them with your classmates. If you still do not understand them, don't worry. You'll come back to them in future lessons.

Lesson 2

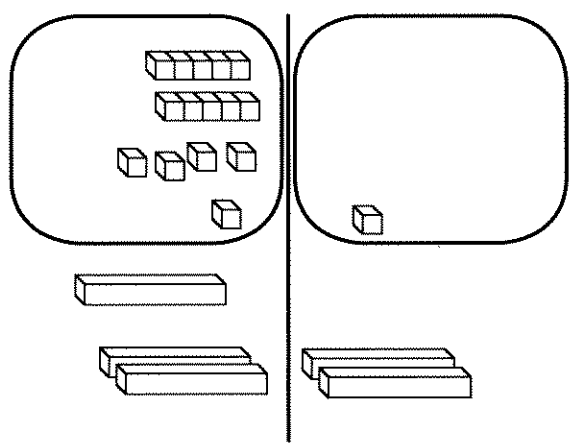
Solving Equations

To solve the equation $3(y - 5) = 2y - 1$ with the Lab Gear, we must have some way to show the multiplication on the left side. This figure shows how.



But to solve the equation, it would be easier to move the upstairs 5-blocks into the minus area, and then reorganize the blocks so matching blocks are near each other.

1. Write the value of *y* that makes both sides equal. (Hint: Start by marking matching blocks.)



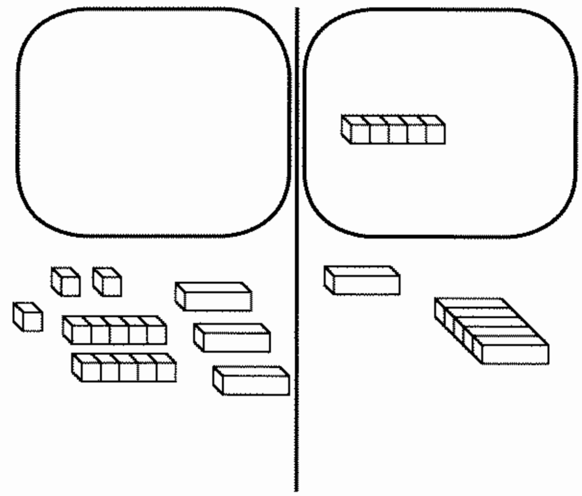
Use the Lab Gear to solve these equations.

2. $3(x + 2) = 15$
3. $2(2x + 5) = 5(x + 1)$
4. $4(2x + 1) = 5(x + 2)$
5. $4(2x - 1) = 5(x + 4)$
6. $3(5 - x) = 9(x - 1)$

Solving Techniques

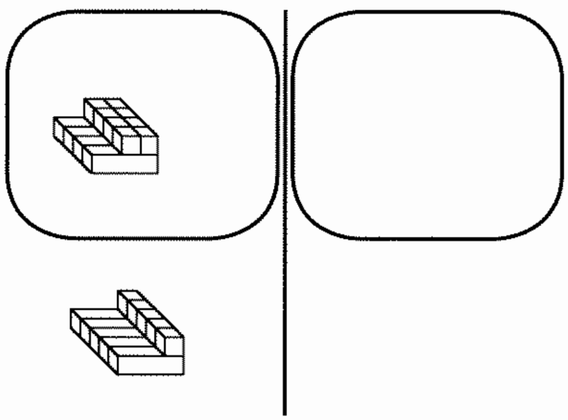
One key to solving linear equations is a technique based on this fact: If two quantities are equal, and you *add the same quantity to both or subtract the same quantity from both*, you end up with equal quantities. This provides you with a method for simplifying equations.

1. Write the equation shown by this figure.
   - Remove three x-blocks from each side.
   - Add 5 to each side and cancel.
   - Finally, form rectangles on both sides.
2. Write the solution to the equation.



For each figure, write the equation, then solve for x. Use the method shown above.

3.



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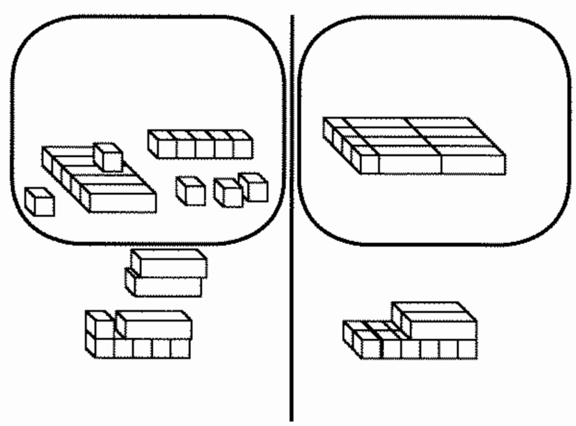
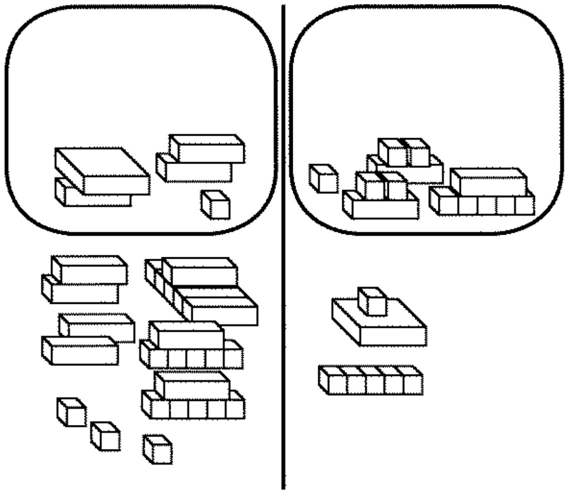


☑ Self-check

3. $x = -5$

5.



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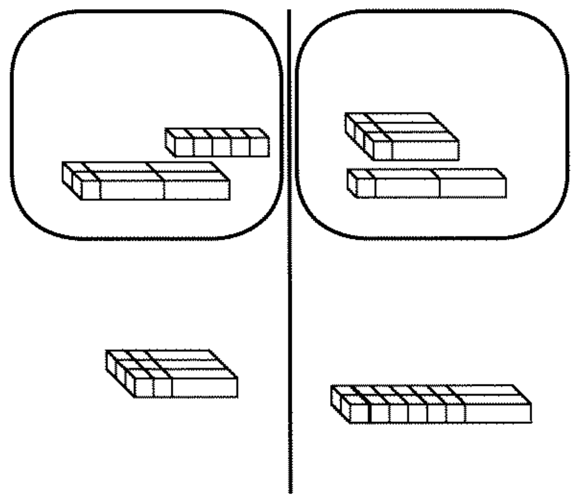
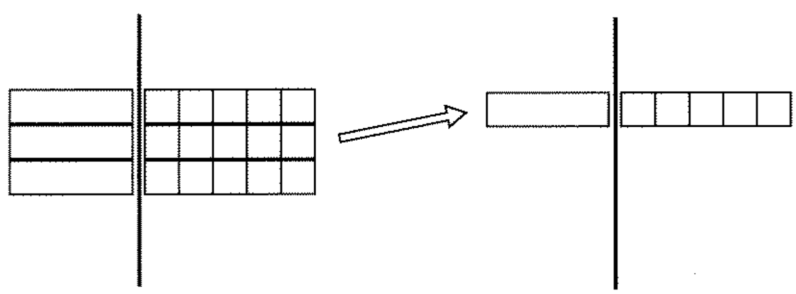


Another key to solving equations is the fact that you can *multiply or divide both sides by the same number* (as long as it's not zero).

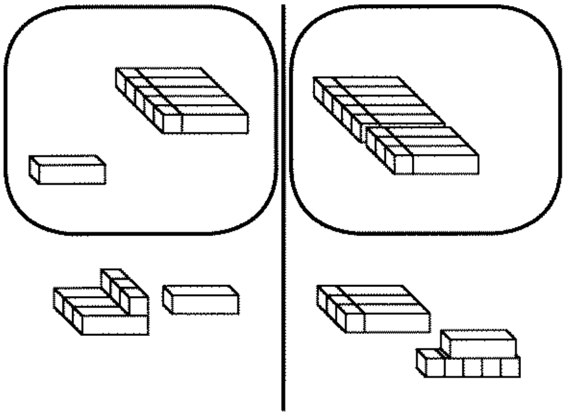
For example, if $3x = 15$, then divide both sides by 3, and you find that $x = 5$.



Of course, some divisions cannot be shown easily with the blocks. If you end up with $4y = 7$, then dividing both sides by 4 will reveal that $y = \frac{7}{4}$. This is impossible to show with the Lab Gear.

Write and solve these equations.

7.



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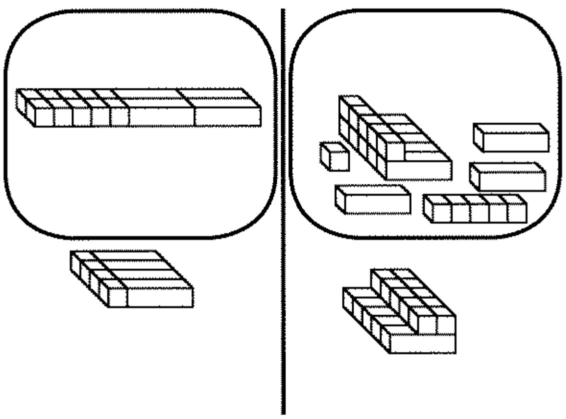

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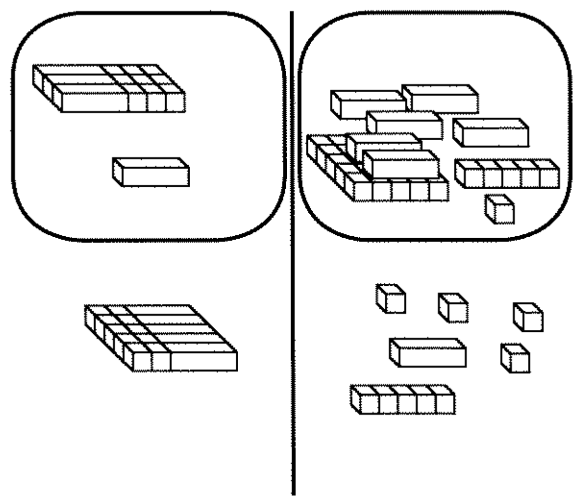

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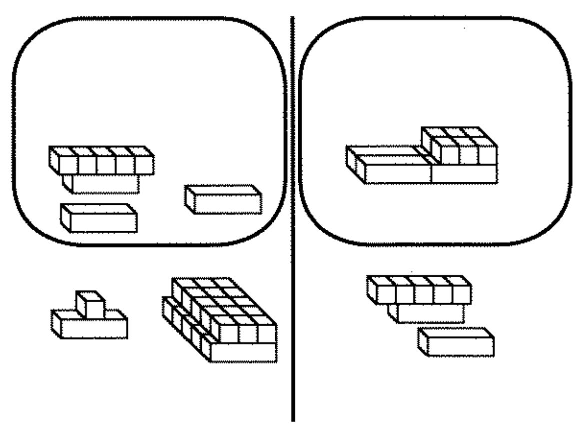

Solving Tricks

Many students discover tricks for solving equations with the Lab Gear. The figures show two simple equations. For each one, solve it, and explain any tricks or shortcuts that you used. Such tricks, if they are correct mathematically, can be used to simplify more complicated problems.

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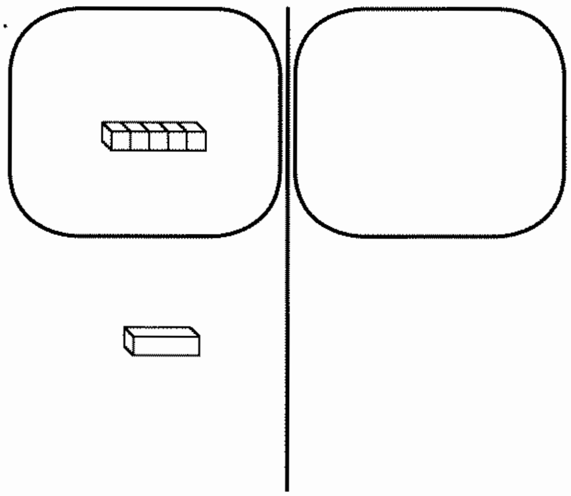

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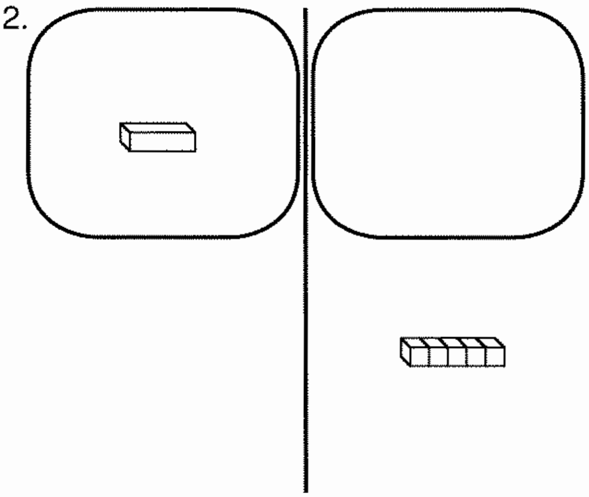
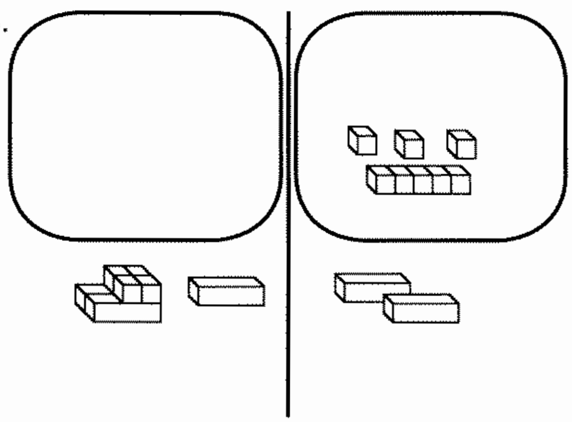


Use all the techniques and tricks you have learned so far to solve these equations. They may have no solutions, one solution, two solutions, or be identities.

3.



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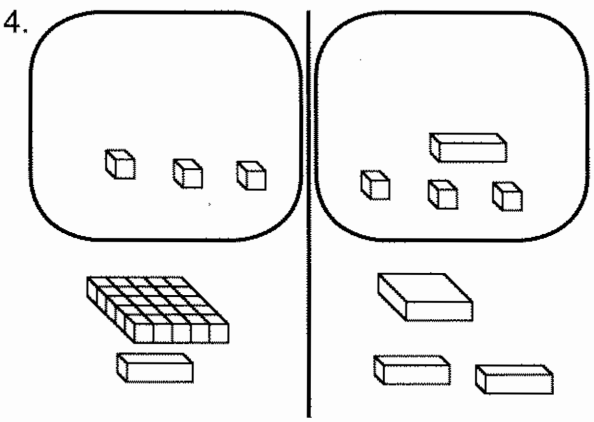

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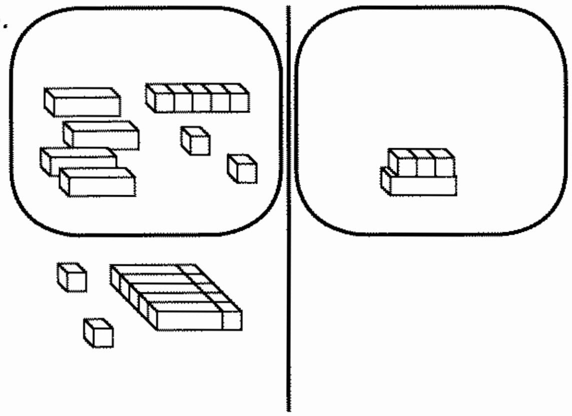

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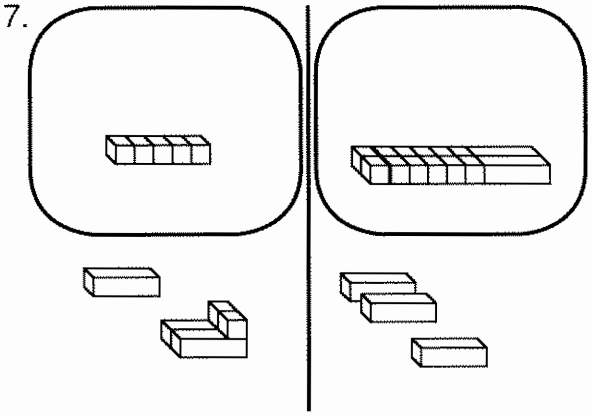

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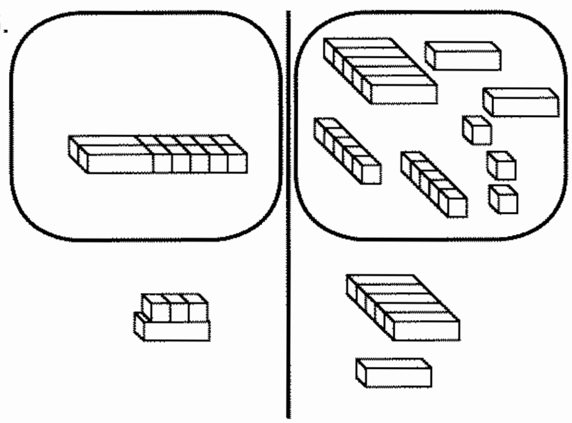

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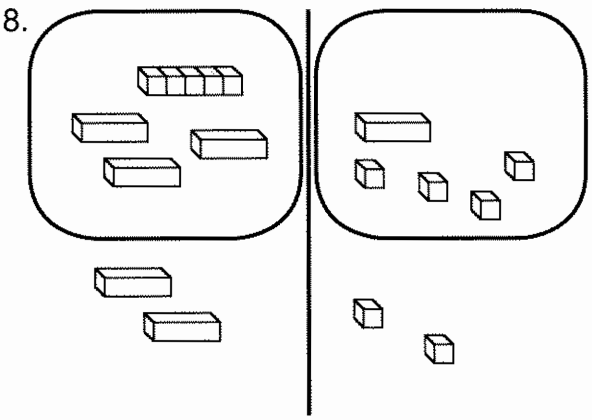

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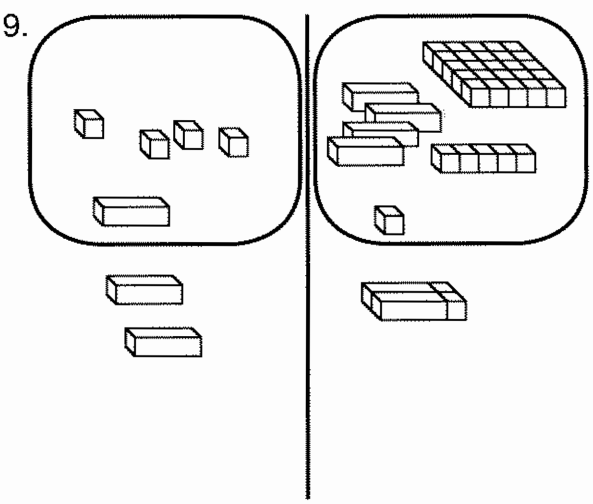


10. Were there any problems in the *How Many Solutions?* exploration at the beginning of this chapter that you couldn't solve? Try them again now.

Equations and Numbers

The first numbers people used were whole numbers. It took many centuries to discover more and more types of numbers. The discovery of new kinds of numbers is related to the attempt to solve more and more equations. The following equations are examples.

a. $x + 2 = 9$
b. $x + 9 = 2$
c. $2x = 6$
d. $6x = 2$
e. $x^2 = 9$
f. $x^2 = 10$
g. $x^2 = -9$

1. Pretend you only know about the **natural numbers**. (These are the positive whole numbers.) List the equations above that can be solved.
2. Pretend you only know about the **integers**. (These are positive and negative whole numbers, and zero.) List the equations above that can be solved. Find one that has two solutions.
3. Pretend you only know about the **rational numbers**. (These are all fractions, positive, negative, and zero. Of course, integers are included, since for example $3 = \frac{6}{2}$.) List the equations above that can be solved.

To solve equation (f), you need a number whose square is 10. The square root key of a calculator provides one answer: 3.16227766. But if you try to multiply this number by itself, you will find that the answer is not exactly 10.

Your calculator should say 9.999999999 for the product of 3.16227766 times 3.16227766. (A powerful computer, or an exceptionally patient and accurate student, would give the answer 9.999999998935076.) That means that 3.16227766 is very, very close to the square root of 10, but it is not exactly the square root of 10.

In later math classes, you will learn that the square root of 10 is not a rational number. It is an **irrational real number**. It is on the number line somewhere between the two rational numbers 3.16227766 and 3.16227767.

To solve equation (g), you need to get off the number line! The solution is a **complex number**, and it is written $3i$. The number i is a number one unit away from 0, but off the number line. $i^2 = -1$. You will learn more about i in an advanced algebra class.

Exploration 3 Make a Rectangle

1. Make as many different rectangles as you can with x^2, $10x$, and any number of yellow blocks. For each one, write a multiplication equation to show that area = length · width.

   Is one of the rectangles a square?

2. Make as many different rectangles as you can with x^2, 18, and any number of x-blocks. For each one, write a multiplication equation to show that area = length · width.

   Is one of the rectangles a square?

Exploration 4 Make a Square

To make a square with these blocks, add as many yellow blocks as you want, but nothing else. For each square, write an equation relating the side length to the area.

1. $x^2 + 10x + \ldots$
2. $4x^2 + 8x + \ldots$
3. $9x^2 + 6x + \ldots$
4. $x^2 + 2x + \ldots$
5. $4x^2 + 12x + \ldots$
6. Is it possible to get a different square by adding a different number of yellow blocks? Explain your answer.

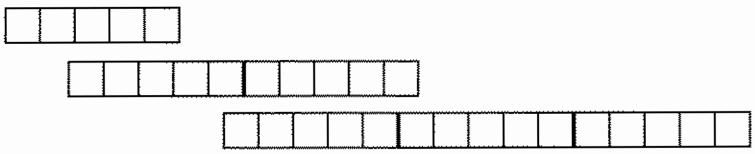
To make a square with these blocks, add as many x-blocks as you want, but nothing else. For each square, write an equation relating the side length to the area.

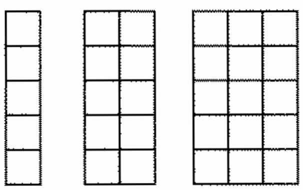
7. $x^2 + \ldots + 25$
8. $4x^2 + \ldots + 25$
9. $x^2 + \ldots + 36$
10. $9x^2 + \ldots + 1$
11. $x^2 + \ldots + 9$
12. Is it possible to get a different square by adding a different number of x-blocks? Explain your answer.

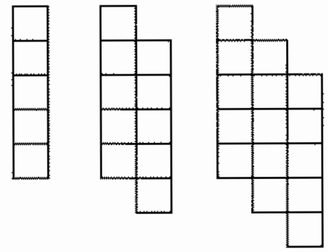
Exploration 5 Perimeter

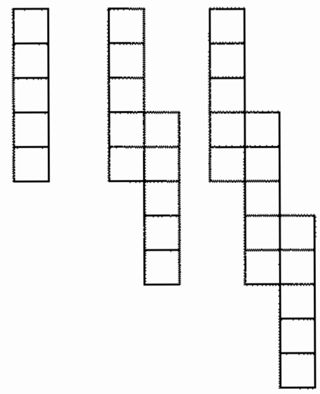
Look at each sequence. Think about how it continues, following the pattern. Write the perimeters of the figures given, then the perimeter of the fourth one, the tenth one, and the hundredth one.

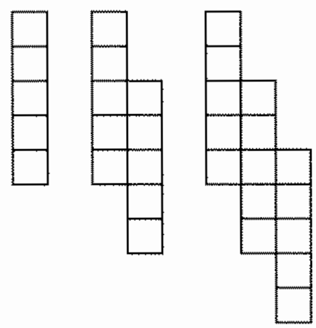
1. 

2. 

3. 

4. 

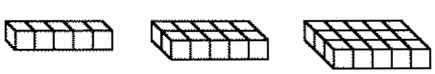
5. 

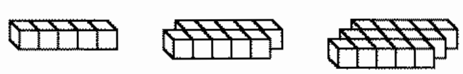
6. 

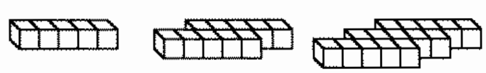
Exploration 6 Surface Area

Look at each sequence. Think about how it continues, following the pattern. Write the surface areas of the figures given, then the surface area of the fourth one, the tenth one, and the hundredth one.

1. 

2. 

3. 

4. 

Chapter 8 Factoring

This chapter motivates the development of factoring techniques by showing students how they can help simplify algebraic fractions and solve some quadratic equations.

New Words and Concepts

The **Zero Product Principle** provides an approach to the solving of quadratic equations. The cancelling of **common factors** is the way to simplify fractions. Both require **factoring**.

While factoring is not an important skill for a physicist or engineer, it does offer useful learning opportunities in an algebra course. First, it constitutes review of the critically important distributive law. Second, because there is no automatic algorithm for factoring, the student learns to use trial and error, and to develop a feel for algebraic manipulation in a puzzle-solving context. Finally, skill at factoring is quite useful in further schoolwork and test-taking.

Teaching Tips

The "Make a Rectangle" Explorations in previous chapters laid the foundation for these lessons. However, they generally limited themselves to binomials with only plus signs. What is new in this chapter, aside from the application of factoring to other problems, is the extension to more complicated cases that involve one or two minus signs. These are quite challenging to work out with the Lab Gear, and some students prefer to do them just by algebra. On the other hand, many students find that there is something exciting about the way the blocks fit perfectly, and the rectangle model holds up even with minus signs.

Lesson Notes

- **Lesson 1,** The Zero Product Principle, page 98: Be sure to precede this lesson with Exploration 1. Note that the equations are given in factored form.
- **Lesson 2,** Reducing Fractions, page 100: Solid understanding of this material, is of course, only possible if one already understands fractions in arithmetic. However, the concrete image of the rectangles with a matching side can go a long way towards clarifying the concept of a common factor, and students who had trouble with simplifying nonalgebraic fractions may be helped significantly by this approach.
- **Lesson 3,** Common Factors, page 102: This lesson is intended to start the work on factoring by defining the terms and the general strategy, while at the same time posing some difficult questions.
- **Lesson 4,** Recognizing Identities, page 104: This is based on the work done in Chapter 6, Lesson 4, as well as various explorations.
- **Lesson 5,** More Factoring, page 106: Straightforward "Make a Rectangle" problems.
- **Lesson 6,** More Difficult Factoring, page 107: Minuses in both factors.
- **Lesson 7,** Even More Difficult, page 108: A minus sign in one of the two factors.
- **Lesson 8,** Factoring Practice, page 109, and **Lesson 9,** Reducing Fractions, page 110: In these lessons, students need to use all the factoring techniques they have learned. Chapter 9, Lesson 2 offers more opportunities for general review of factoring, in the context of solving equations.

Exploration 1 True or False?

Suppose a and b are numbers. Indicate whether these statements are true or false by writing T or F. Explain each answer.

1. If $a = 0$ and $b = 0$, then $ab = 0$
2. If $a \neq 0$ and $b = 0$, then $ab = 0$
3. If $a = 0$ and $b \neq 0$, then $ab = 0$
4. If $a \neq 0$ and $b \neq 0$, then $ab = 0$
5. Change one symbol in statement 4 to make it true. (Do not change it to any of the statements 1-3.)
6. If you know that $ab = 0$ what can you conclude about a and b?

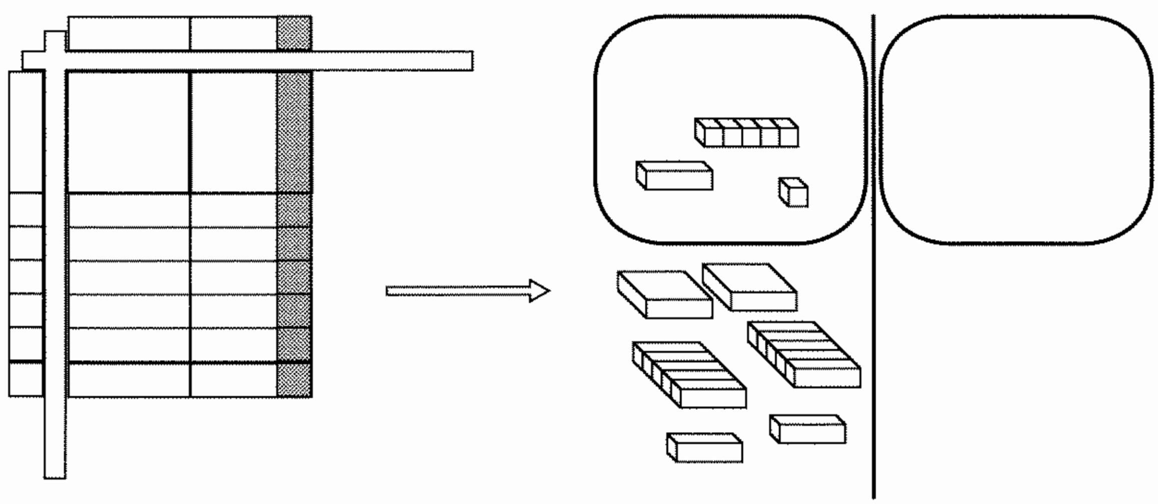
Lesson 1

The Zero Product Principle

Consider using blocks to solve the equation $(x + 6)(2x - 1) = 0$

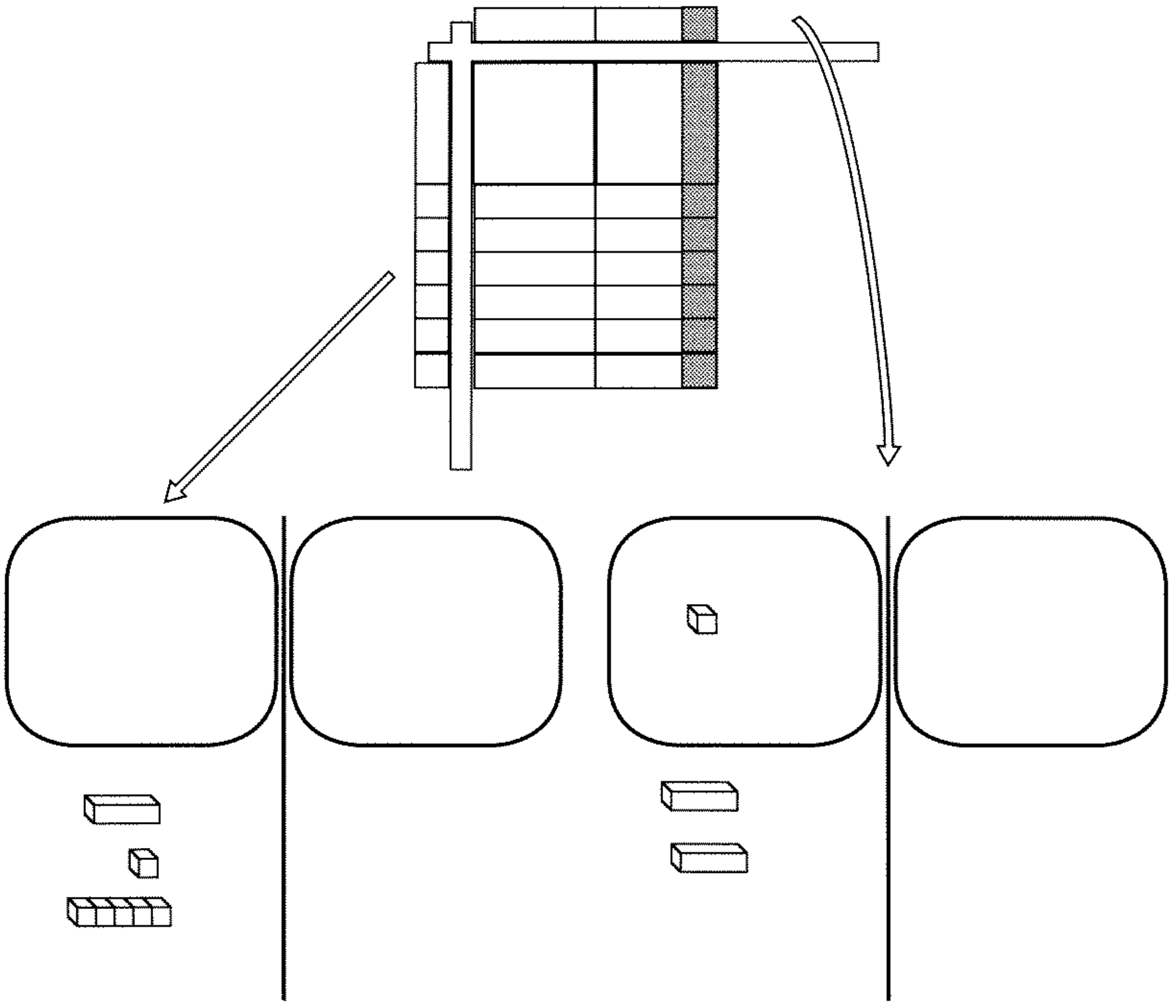
Setting this up with the Lab Gear gives this figure.

1. Explain the figure.



If you try to solve this equation with the methods you learned in chapter 7, you will find that it is not possible. For an equation of this type, we need a whole new approach—one based on the **Zero Product Principle**, which says: *When the product of two quantities is zero, one or the other quantity must be zero.*

Remember, the equation we are trying to solve is $(x + 6)(2x - 1) = 0$. Since the product is zero, you can write these two equations.



$x + 6 = 0$ or, $2x - 1 = 0$

2. You know how to solve these equations. Write the solutions.
3. There are two solutions to the equation $(x + 6)(2x - 1) = 0$. What are they?

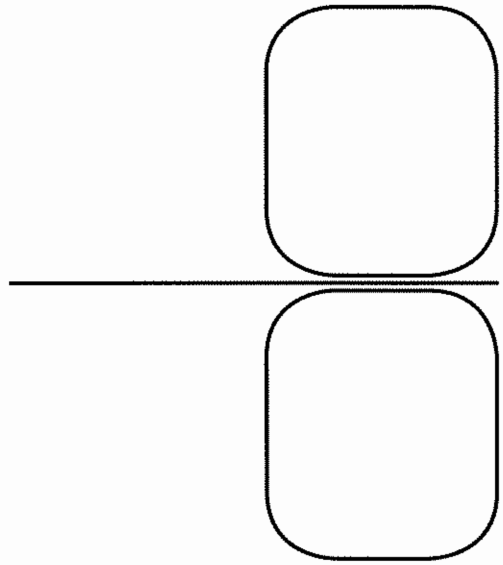
Solve these equations.

4. $(3x + 1)(x + 5) = 0$
5. $(2x + 3)(5 - x) = 0$
6. $(2x - 2)(3x - 1) = 0$

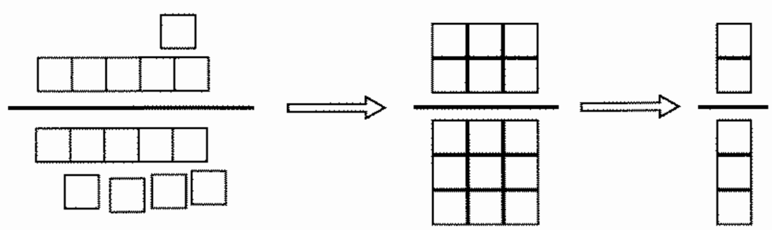
Reducing Fractions

As you know, to reduce fractions, you divide the numerator and denominator by a common factor. For example, $\frac{6}{9}$ can be reduced by dividing 6 and 9 by 3, which gives $\frac{2}{3}$.

This can be modeled with the Lab Gear. To do that, use the workmat turned on its side. Instead of representing an equals sign, the middle line now represents the fraction bar.



Arrange the blocks in the numerator into a rectangle (or square). Do the same in the denominator. If the two rectangles share a side, they can be lined up on either side of the line. Dividing *top and bottom* by the common side, will give you the new numerator and denominator.



Reduce these fractions by making rectangles with your Lab Gear.

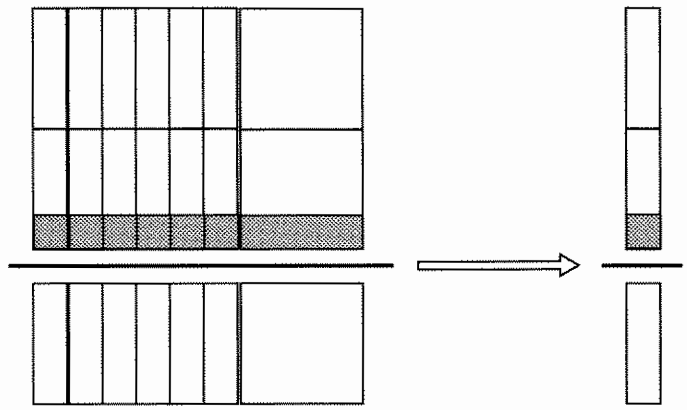
1. $\frac{4}{12}$

2. $\frac{8}{24}$

3. $\frac{12}{18}$

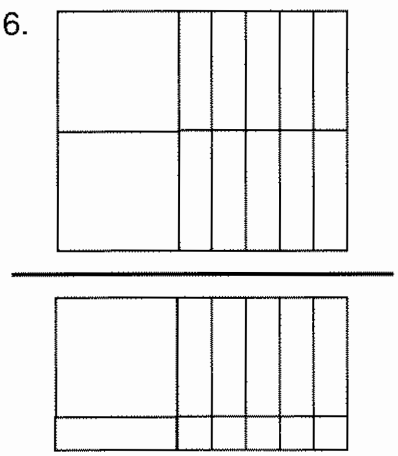
Using the same method, you can reduce algebraic fractions.

4. Name the fraction that is being reduced in this figure.
5. Name the reduced fraction.



Write and reduce these fractions.

6.



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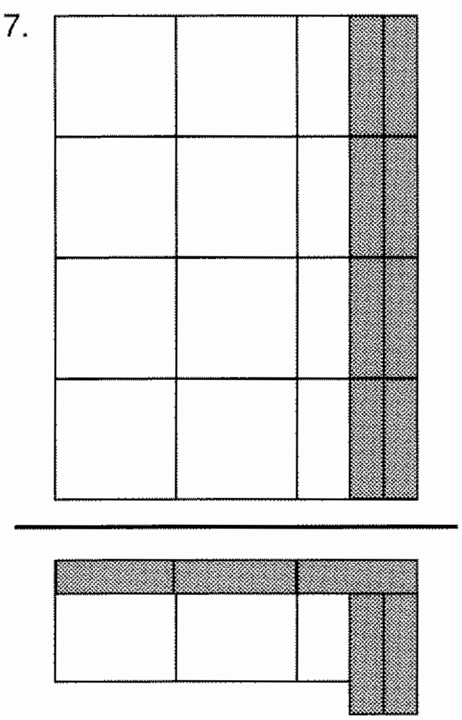

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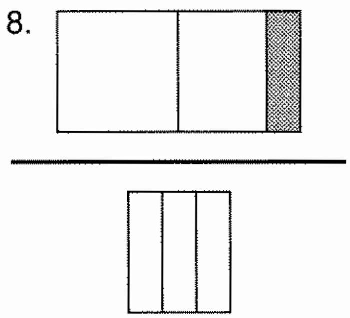


Common Factors

So far in this chapter, you have learned how to solve equations using the Zero Product Principle and you have also learned how to reduce algebraic fractions.

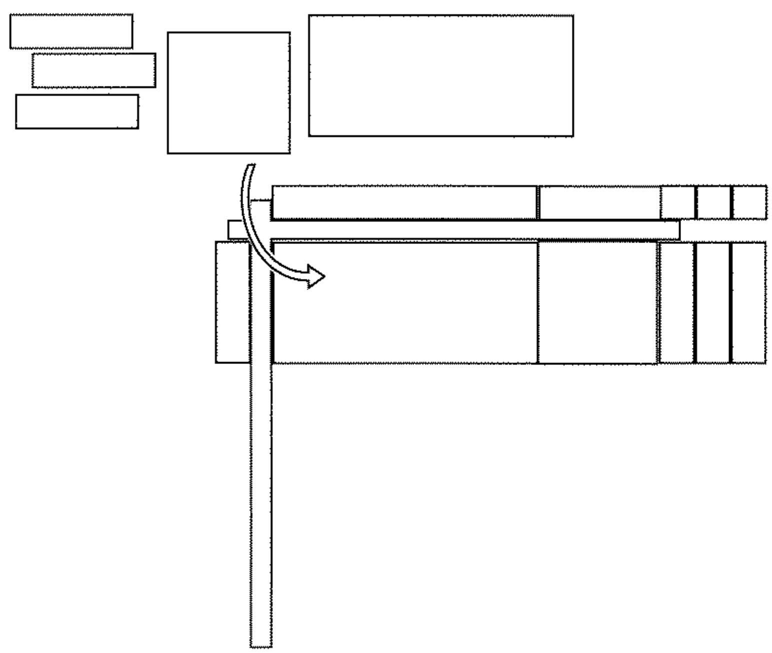
In both cases, using the Lab Gear, the blocks must be arranged in rectangles (or squares). To reduce fractions, you must have a rectangle above the fraction bar, and a rectangle below. To solve equations using the Zero Product Principle, you need a rectangle that is equal to zero.

Unfortunately, blocks are not usually arranged in a rectangle. (In fact, it is often impossible to arrange them that way.) In the rest of this chapter, you will learn some more ways to rearrange blocks into rectangles.

Arranging Lab Gear blocks into rectangles is the same as rewriting polynomials as products. In algebra this process is called **factoring**.

A *factor* of an algebraic expression is an expression that divides it evenly. For example, x is a factor of x^2, since $x \cdot x = x^2$.

1. Write the polynomial (inside the corner piece) that is being factored.
2. Write the two factors of the polynomial.



If you multiply the two factors, using the distributive rule, you would get the original polynomial back. Factoring is applying the distributive rule in reverse.

Use the Lab Gear to factor these polynomials. Not all are possible.

3. $2x^2 - x$
4. $2x^2 + 6x + 1$
5. $x^2 + 2x + xy$
6. $3x^2 - 3x$

In problem 5, x is a factor of x^2, of $2x$ and of xy because it divides each term evenly. We say that x is a *common factor* of the three terms.

The following problems are more challenging. Even though there is no common factor, the blocks can be rearranged into a rectangle (or square). Remember that it is the *uncovered* part of the blocks that needs to make a rectangle.

7. $x^2 + 6x + 9$
8. $y^2 - 2xy + x^2$
9. $y^2 - 4$

If you have trouble, get help from your classmates. Don't worry, you will learn more about how to factor these polynomials in the next lesson.

Exploration 2 Inequalities

For each equation, find the values of x that make it true.

1. $x + 5 > 1$
2. $x - 5 > 1$
3. $x + 6 > 0$
4. $x - 6 > 0$
5. $x - 1 > 5$
6. $x + 1 > 5$
7. $x - 5 > -1$
8. $x + 5 > -1$
9. $-x > 6$

10. Check your answer to problem 9 by carefully trying various values for x. (Most algebra students get this problem wrong, and come up with the answer $x > -6$.)

11. For what values of x is $-x > -6$?

12. Explain how problems 9 and 11 differ from problems 1-8. Tell how to handle this situation when trying to solve an inequality.

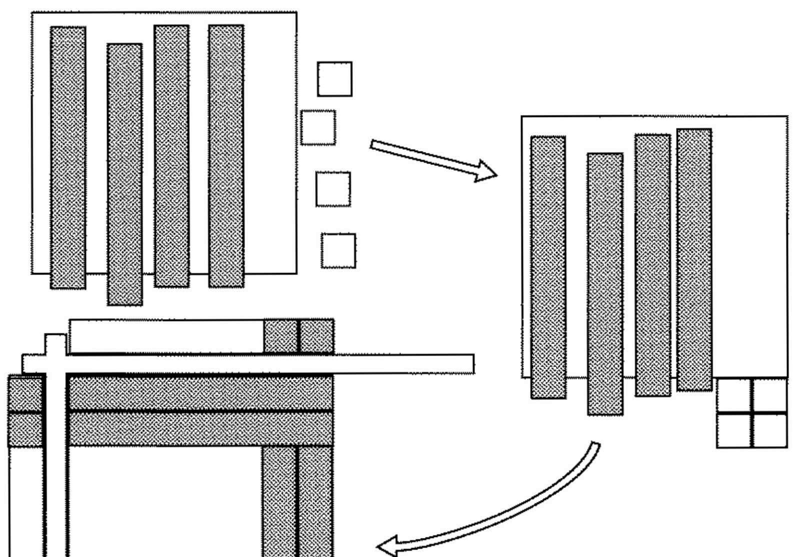
Recognizing Identities

When there is no common factor, it is still sometimes possible to rearrange the blocks into a rectangle (or square). In these problems, factor the polynomials by arranging the blocks into a *square*. Not all are possible.

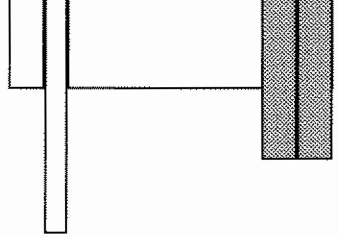
1. $x^2 + 4x + 4$
2. $9x^2 + 6x + 1$
3. $x^2 + 4x + 9$
4. $x^2 + 6x + 3$
5. $x^2 + 2xy + y^2$
6. $2x^2 + 8x + 16$
7. Explain what makes some of these problems possible, and others not.

These problems were based on the identity $(a + b)^2 = a^2 + 2ab + b^2$, which you used in Chapter 6. It is a little more difficult when there are minuses.

Look at this example.



- This figure shows how to set up the downstairs blocks first.



- Next, rearrange the upstairs blocks. Notice that the uncovered area is a square.

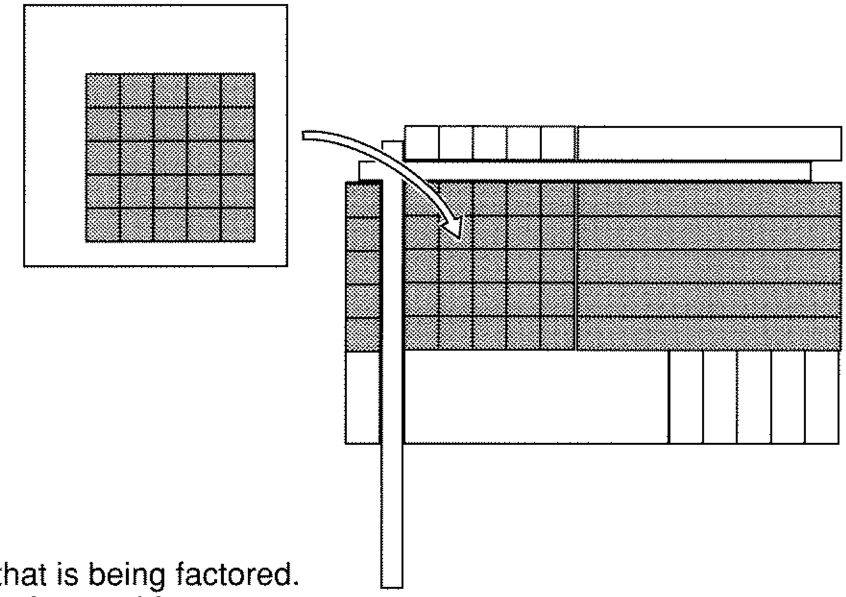
8. Write the polynomial that is being factored.
9. Write the two factors of the polynomial.

Use the same method to factor these polynomials. Not all are possible.

10. $4y^2 - 4y + 1$
11. $y^2 - 3y + 9$
12. $y^2 - 6y + 9$
13. $y^2 - 10y + 25$
14. Each of the polynomials in problems 1, 2, and 5 is the *square of a sum*. How would you describe each of the polynomials in problems 10, 12, and 13? Write this identity in terms of variables *a* and *b*.

This example shows a type of factoring based on the third important identity—the *difference of two squares*.

In this case, the only way to get a rectangle is to *add zero*, with $5y$ and $-5y$, and to arrange the blocks as shown. The uncovered rectangle has dimensions $y + 5$ by $y - 5$.



15. Write the polynomial that is being factored.
16. Write the polynomial in factored form.

Use this method to factor these polynomials.

17. $x^2 - 1$
18. $y^2 - 4$
19. $y^2 - x^2$
20. Go back to problems 7-9 in Lesson 3, page 103. If you had trouble factoring these expressions, try them again.

Factor these polynomials. Use any of the methods you have learned. Not all are possible.

21. $4x^2 + 4x + 1$
22. $x^2 + 2x$
23. $y^2 - 9$
24. $y^2 - 8y + 16$
25. $x^2 + 9$

Solving Equations

Use any method to solve these equations. Remember the Zero Product Principle.

26. $y^2 - 16 = 0$
27. $5x + 2 = x - 6$
28. $2x^2 + 6x = 0$
29. $4x^2 + 12x + 9 = 0$
30. $y^2 - 8y + 16 = 0$

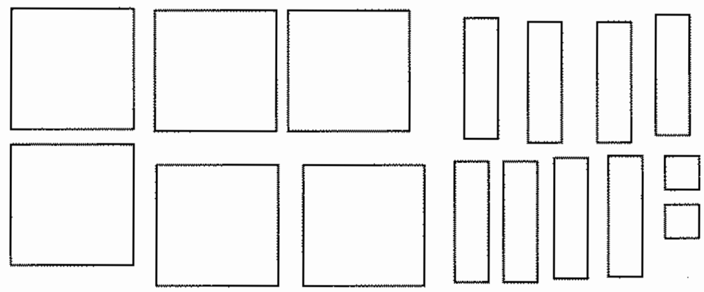
☑ Self-check

21. $(2x + 1)^2$

More Factoring

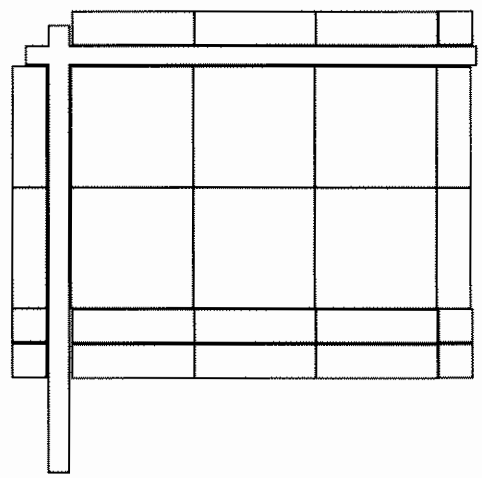
You have learned to factor polynomials that have a common factor, or that are examples of a standard identity. Some other polynomials can be factored by using *trial and error.*

Look at this example. Show it with your blocks. Try to find the factors without looking at the solution below. Use trial and error.



Check your work with this figure.

1. Write the polynomial that is being factored.
2. Write the polynomial in factored form.



Using trial and error, factor these polynomials. One is impossible.

3. $9x^2 + 6x + 1$
4. $2x^2 + 5x + 2$
5. $x^2 + 7x + 10$
6. $3x^2 + 4x + 2$
7. $2x^2 + 11x + 5$

☑ Self-check

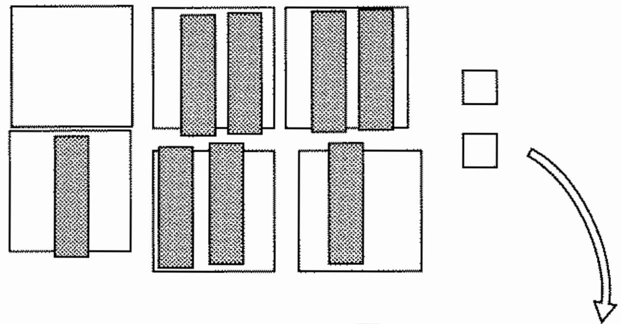
4. $(2x + 1)(x + 2)$

More Difficult Factoring

As you might guess, minus signs make it more difficult to factor polynomials.

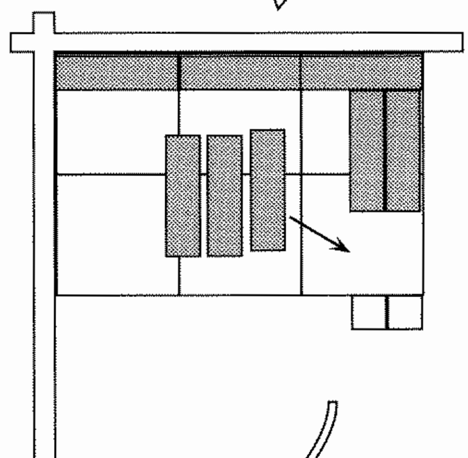
Look at this figure. Show it with your blocks.

1. Write the polynomial that is being factored.



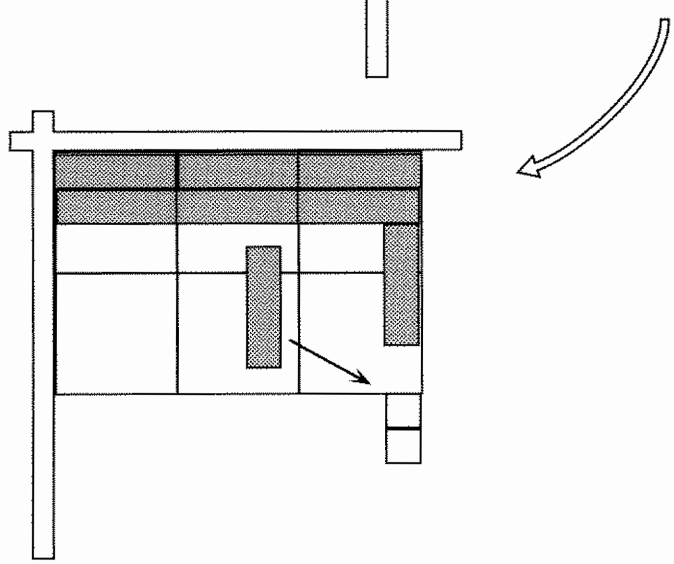
- First arrange the downstairs blocks. Remember that the goal is to get the uncovered part to be a rectangle.

The first try shown in this figure does not work, because even after sliding two more x-blocks to the bottom right, there would be one left over, ruining the rectangle.



- However, by rearranging the yellow blocks and the upstairs x-blocks, it is possible to solve the problem.

2. Write the polynomial in factored form.



Using this method, factor these polynomials.

3. $y^2 - 6y + 5$
4. $y^2 - 2y + 2x - xy$
5. $4x^2 - 12x + 5$
6. $3x^2 - 5x + 2$

☑ Self-check

3. $(y - 1)(y - 5)$

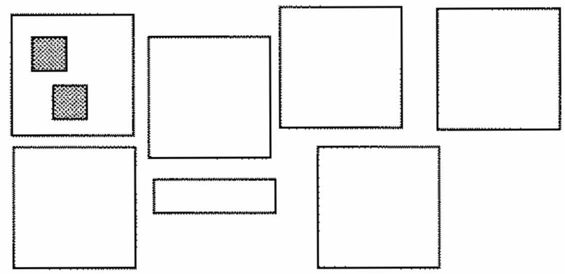
Even More Difficult

1. Multiply $(2x - 2)(3x + 1)$ with the Lab Gear. Notice that at the end, you can cancel matching upstairs and downstairs blocks ($2x$ and $-2x$). Write the answer. Now, reverse this canceling to factor the answer.

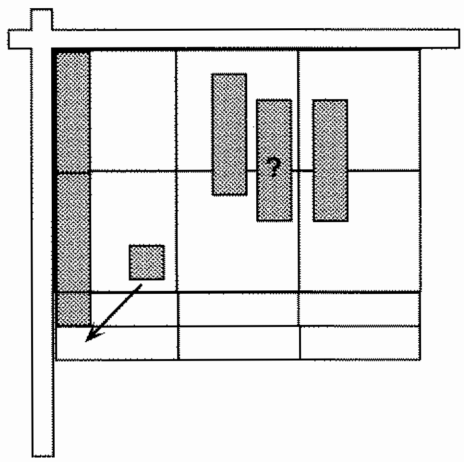
When trying to factor polynomials with the Lab Gear, it is sometimes necessary to "uncancel" upstairs and downstairs blocks, or once again, to *add zero*.

Look at this example. Show it with your blocks.

2. Write the polynomial that is being factored.

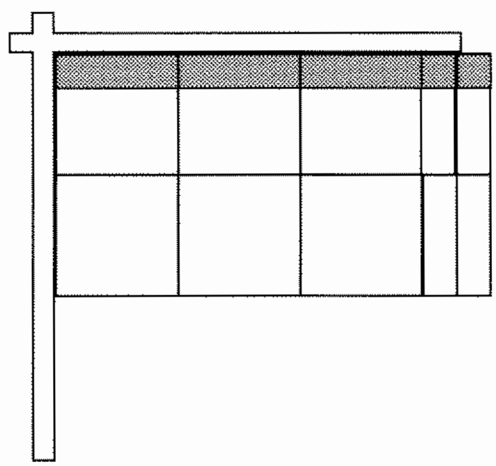


The first try at factoring, involving an additional $5x$ and $-5x$ does not work, since we have three upstairs x-blocks left which prevent us from getting an uncovered rectangle.



To find the solution we must add $3x$ and $-3x$.

3. Write the factors of the polynomial.



For these expressions, multiply, cancel, and write the answer. Then *uncancel* and factor. In other words, recreate the rectangle.

4. $(y - 3)(y + 1)$
5. $(2x - 2)(x + 5)$

Using this method, factor these polynomials.

6. $3x^2 + x - 10$
7. $2x^2 + 2x - 12$
8. $4x^2 - 4x - 3$
9. $y^2 - xy - 5x + 5y$

Factoring Practice

Factor these polynomials. One is impossible.

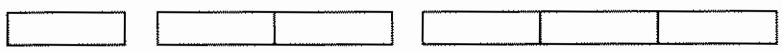
1. $xy + 6y + y^2$
2. $y^2 - 16$
3. $3x^2 + 13x - 10$
4. $4x^2 + 8x + 4$
5. $2x^2 + 2x + 1$
6. $y^2 - 5y + 6$
7. $y^2 - 4y + 4$
8. $x^2 + 8x + 12$

☑ Self-check

3. $(3x - 2)(x + 5)$

Exploration 3 Perimeter

Look at each sequence. Think about how it continues, following the pattern. Write the perimeters of the figures given, then the perimeter of the fourth one, the tenth one, and the hundredth one.

1.


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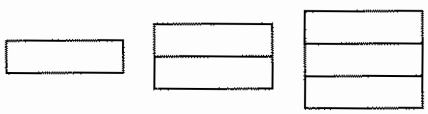
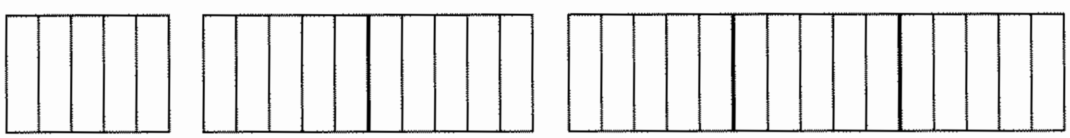


3.


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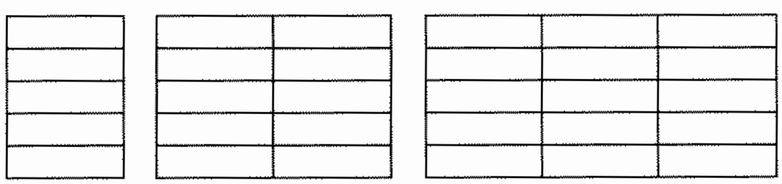

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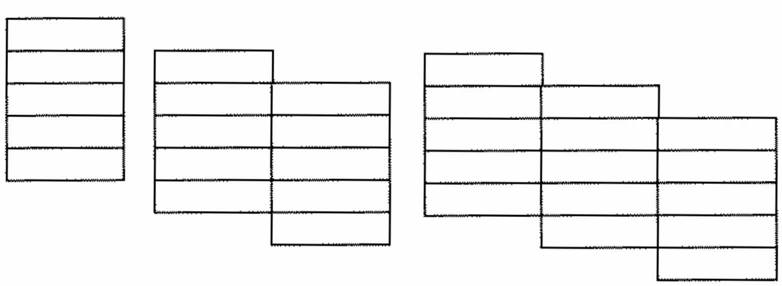
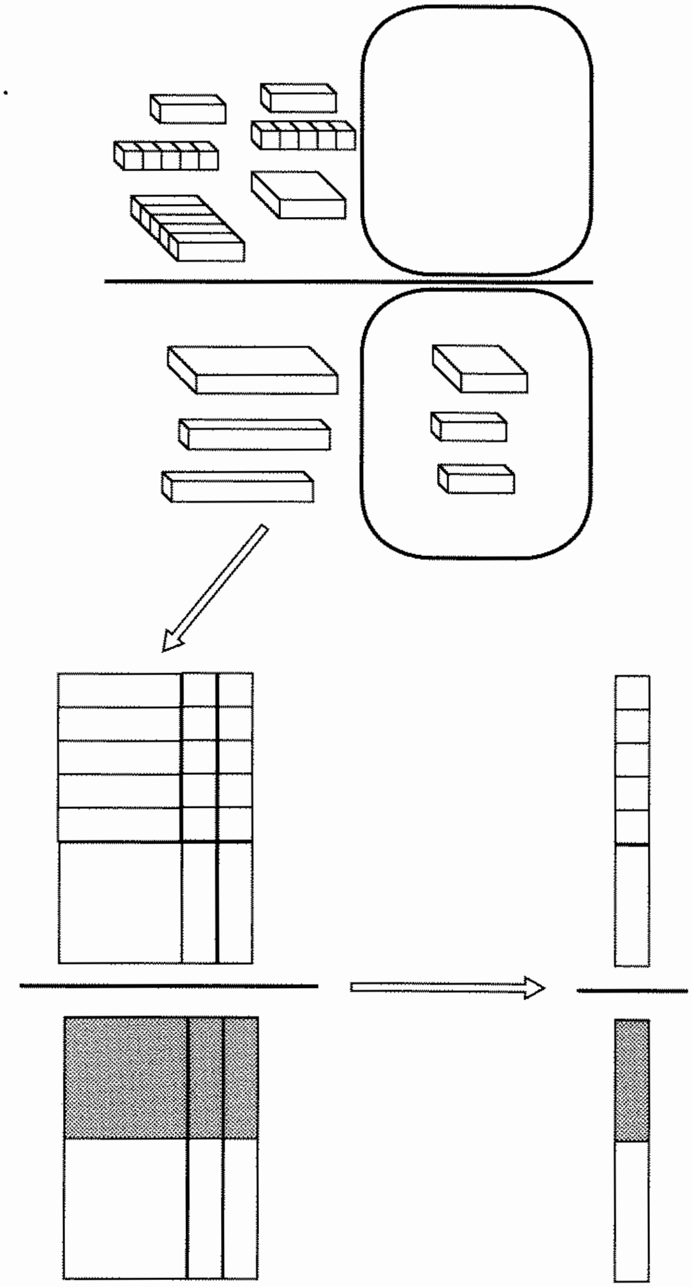


Reducing Fractions

To reduce fractions, remember that you can use the Lab Gear to make rectangles that match in one dimension.

1. Write the fraction that is being reduced.
2. Write the fraction in its factored form.
3. Write the reduced fraction.



Use this method to reduce these fractions. One is impossible.

4. $\dfrac{5y-5x}{y^2-x^2}$

5. $\dfrac{x^2+3x+2}{x^2-4}$

6. $\dfrac{x^2+5x}{2x^2+xy}$

7. $\dfrac{y^2-7y+12}{y^2-6y+9}$

8. $\dfrac{x+5}{2x^2-5x-25}$

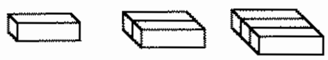
☑ Self-check

7. $\dfrac{y-4}{y-3}$

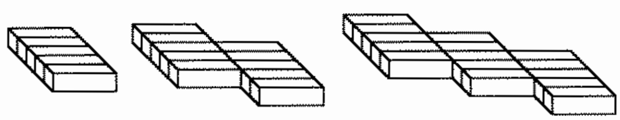
Exploration 4 Surface Area

Look at each sequence. Think about how it continues, following the pattern. Write the surface areas of the figures given, then the surface area of the fourth one, the tenth one, and the hundredth one.

1. 

2. 

3. 

4. 

Chapter 9 More Equation Solving

This chapter extends what was learned in chapters 7 and 8 by using those techniques to solve more challenging equations.

New Words and Concepts

The Lab Gear techniques for solving **simultaneous equations** are a natural extension of the work done in solving linear equations. Substitution is really the only convenient Lab Gear method, and you will need to teach the other methods (graphic solutions and linear combinations) some other way. Nevertheless, the exercises in this section are very valuable, as they establish some basic concepts about simultaneous equations.

As for quadratics, after reviewing factoring, this chapter builds on past "Make a Square" activities, to lead the students to learn about **completing the square**. This lays the foundation for a solid understanding of the derivation of the quadratic formula, the traditional climax of first year algebra.

Teaching Tips

The final lessons are all quite challenging, and are a way to bring together everything that was learned throughout the Algebra Lab program. This would be a good time to work on any unresolved "Always, Sometimes, or Never?" questions that may be left over from previous chapters.

Lesson Notes

- **Lesson 1,** Simultaneous Equations, page 114: It requires two workmats to work each of these problems, so make sure you have duplicated enough copies.
- **Lesson 2,** Zero Products, page 118: A chance to review material from Chapter 8.
- **Lesson 3,** Equal Squares, page 118: Another approach in the first example is to subtract 25 from both sides, and then factor the left side. That method, based on the difference of squares identity can be used for all "equal squares" problems.
- **Lesson 4,** Completing the Square, page 120: Make sure the students remember that "equal squares" problems usually have two solutions.

Selected Answers

Chapter 1

Lesson 1, pages 3-4

5. 1 • 30, 2 • 15, 3 • 10, 5 • 6
7. See page 1 for a complete set of labeled blocks.
8. $5x = 30$ and $x^2 = 36$
9. $5x = 31$ and $x^2 = 38.44$

Lesson 3, page 6

1. a. 9
   b. 17
   c. 3
   d. 11.19
2. a. 25
   b. 16
   c. 13
   d. 20
3. a. 70
   b. 25
   c. 19.4
   d. $29\frac{1}{2}$

Lesson 4, page 8

1. $4y$
2. $x + 4$
3. $y + 2x$
4. $3x^2 + 2x + 6$
5. $2y^2 + 2x + 3$
6. $x^2 + 3x + 3$
7. $3x^2 + 2y + 5$
8. $3x^2 + 2x + 10$
9. $3x^2 + +2xy + y$

Lesson 5, page 9

1. negative 3
2. subtract 2 from 5, 3
3. the opposite of negative 5, or 5
4. subtract 7 from 3, negative 4
5. the opposite of $(4 + \frac{1}{2})$, negative $4\frac{1}{2}$
6. the opposite of 6 subtracted from 2, 4
7. x subtracted from 3
8. the opposite of negative x, x
9. y
10. the opposite of $(4x + 1)$
11. 5 subtracted from y
12. the opposite of x subtracted from y

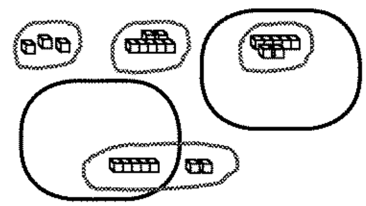
Exploration 1, page 9

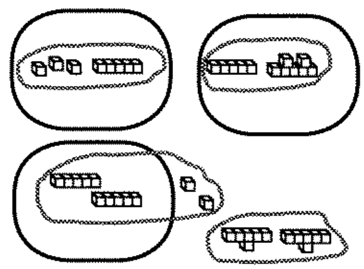
1. a. –2
   b. 3
2. a. false
   b. true

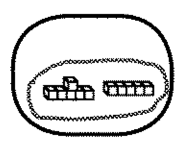
Lesson 6, pages 10-11

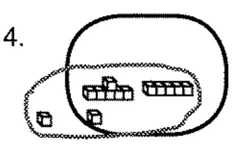
1. 4
2. –3
3. 19
4. –1
5. 2
6. –15
7. –1
8. –3
9. 13
10. –15

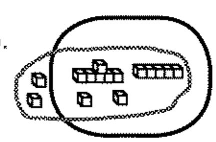
Exploration 2, page 14

1. 

2. 

3. 

4. 

5. 

Chapter 2

Lesson 1, pages 16-17

1. No, the order doesn't matter.
2. $-6 + (-3) = -9$
3. $25 + 4 + (-12) + (-1) = 16$
4. –3
5. –18
6. 1
7. 6
8. –20
9. –18
10. 5
11. 18
12. 0

Lesson 2, page 20

1. –4
2. 4
4. –10
5. –2
6. –11
7. 13
8. –14
9. 5
10. 9
11. –5
12. 10
13. –14
14. –23
15. –20
16. 10
17. –11
18. 9

Lesson 3, page 22

3. Changing the order of the numbers does not affect the answer.
4. –8
5. –6
6. 20
7. 12
8. –20
9. –21
10. –3
11. 6
12. –5
13. 6
14. multiplication
15. 2
16. –3
17. –4
18. 5
20. Blocks inside the minus area move outside; blocks outside the minus area move inside.

Lesson 4, page 23

5. two squared
6. negative four, squared
7. the opposite of three squared
8. six x squared
9. negative six x, squared
10. the opposite of six x squared
11. $(6x)^2$ and $-(6x)^2$ are equal
12. 2 is the exponent, 7 is the base
13. 3 is the coefficient

Exploration 1, page 23

1. P
2. P
3. N
4. P or 0
5. P or 0
6. P or 0

Lesson 5, page 24

5. 6.16
6. 3.9
7. –3066.1
8. $-\frac{1}{2}$
9. $-\frac{5}{6}$
10. $-\frac{2}{18}$

Lesson 6, pages 25-26

1. What times –2 equals –8?
2. What times 2 equals –8?
6. a., b., and e. are equal; c. and d. are equal

Exploration 2, page 27

1. area 31, perimeter 32
2. area 31, perimeter 24
3. area 31, perimeter 28

Exploration 3, page 28

1. volume 1, surface area 6
2. volume 25, surface area 70
3. volume 100, surface area 130
4. volume 36, surface area 84
5. volume 36, surface area 94
6. volume 36, surface area 92

Chapter 3

Lesson 1, pages 30-31

1. –5
2. to show –(–3), the opposite of –3, or 3
3. –10
4. a. $-(x + 5) + (x^2 - 1) + 5x$
   b. $x^2 = 4$, blocks show 6
   c. $x^2 = 9$, blocks show –9
   d. –1
5. a. $-(y^2 + y + x) + x^2 + 10 + (xy - x)$
   b. $x^2 = 1$, $y^2 = 4$, $xy = -2$, blocks show 6
   c. 13
   d. –20

Lesson 2, page 32

1. $5 \cdot 7 = 35$
2. $\frac{35}{7} = 5$
3. $\frac{35}{5} = 7$
5. Impossible to build a rectangle with length 0.
6. If $\frac{12}{0} = 0$ was true, then $0 \cdot 0 = 12$ would have to be true.

Exploration 1, page 33

1. length $2x + 12$, width x
2. length $x + y + 1$, width x
3. length $y + x + 5$, width y
4. length $3x + 6$, width x; length $x + 2$, width $3x$
5. length $3x$, width $x + 3$
6. length y, width $3x + 2 + y$
7. Impossible, you can't make a rectangle.
8. length x, width $x + 5$

Lesson 3, page 34

3. $x^2 + x$
4. $x^2 + 5x + 6$
5. $x^2 + 5x + xy + 5y$
6. $2y^2 + 2y$
7. $y^2 + 5y + 4$
8. $2x^2 + 2xy + 5x + 3y + 3$
9. Show three groups of $4x^2 + 6$. The answer is $12x^2 + 18$.
10. Show 5 groups of $y^2 + 2y + 9$. The answer is $5y^2 + 10y + 45$.

Lesson 4, page 35

1. $\frac{(x^2 + 3x)}{x} = x + 3$
2. $x + 4$; $x(x + 4) = x^2 + 4x$
3. $2x + 3$; $3(2x + 3) = 6x + 9$
4. $x + y$; $x(x + y) = x^2 + xy$
5. $2x + 1$; $3x(2x + 1) = 6x^2 + 3x$
6. not possible with blocks
7. $2y + 3$; $2y(2y + 3) = 4y^2 + 6y$
8. $2y + 1$; $y(2y + 1) = 2y^2 + y$
9. $2y + x + 5$; $2(2y + x + 5) = 4y + 2x + 10$
10. make three groups of $2x^2 - 3$
11. make two groups of $2x^2 + 3x - 5$

Lesson 5, pages 36-37

1. a. $-5 > -7$
   b. $-5 < -1$
2. $-6 < 10$
3. $3 < 5$
4. $x = x$
5. $10 = 10$
6. $15 > 3$

Exploration 2, page 38

1. area = $5x$, perimeter = $2x + 10$
2. area = x^2, perimeter = $4x$
3. area = y, perimeter = $2y + 2$
4. area = $5y$, perimeter = $2y + 10$
5. area = y^2, perimeter = $4y$
6. area = xy, perimeter = $2x + 2y$

Lesson 6, pages 40-41

1. $4x - 5 + 2 - (2x - 3)$ and $3x^2 - x^2 + x + 2 - 1 - (2x^2 - 1)$
2. $2x$ and $x + 2$
3. $3x < 3x + 1$
4. $5x = 5x$
5. $3x + 5$ and $4x$; impossible to tell
6. $3x + x^2 + 1 > 3x$ (because $x^2 + 1$ is positive)

Lesson 7, page 42

1. a, b
2. a, c, d
3. c
4. a, c, d

Exploration 3, page 42

1. s.a. = $4x + 2$; $v = x$
2. s.a. = $2x^2 + 4x$; $v = x^2$
3. s.a. = $12x + 10$; $v = 5x$
4. s.a. = $12y + 10$; $v = 5y$
5. s.a. = $2y^2 + 4y$; $v = y^2$
6. s.a. = $2xy + 2y + 2x$; $v = xy$

Lesson 8, page 43

1. 25 is 20 more than 5; 25 is 5 times greater than 5.
2. 6 is 5 more than 1; 6 is 6 times greater than 1.
3. 4 is 2 more than 2; 4 is 2 times greater than 2.
4. 15 is 12 more than 3; 15 is 5 times greater than 3.
5. 42 is 35 more than 7; 42 is 6 times greater than 7.
6. 10 is 0 more than 10; 10 is 1 times greater than 10.
7. 9 is 1 more than 8; 9 is 1.125 times greater than 8.

Exploration 4, page 43
1. always
2. always
3. sometimes ($x = 2$)
4. sometimes ($y = 1$)
5. never
6. always (except $x = 0$)
7. never
8. sometimes ($x = 15$)

Chapter 4

Lesson 1, page 45
1. $-(x^2 + 2x + 12) + 3x^2 + 5x + 8$
2. 10
3. $2x^2 + 3x - 4$
4. 10
5. $-(y^2 - y)$
   $-y^2 + y$
6. $-(25 + x + x^2 - x - 5 - 2x - 1)$
   $- x^2 + 2x - 19$
7. $-(y^2 - y - 2x^2 + 2y + x) +$
   $(2xy + x^2 + 5x + 2x - x - 3)$
   $-y^2 - y + 3x^2 + 5x + 2xy - 3$
8. $-(4x^2 - 2x + 1 - 1) + 10x - x$
   $+ 10 - 1$
   $-4x^2 + 11x + 9$
9. $-(10x - x + 10 - 1) + 4x^2 - 2x$
   $+ 1 - 1$
   $4x^2 - 11x - 9$
10. $-(25 - x^2 + 5) + x^2 - 5x$
    $+ 11 + x$
    $2x^2 - 4x - 19$

Lesson 2, page 47
1. 4
2. no, $9 - (3 + 2) = 4$; $9 - 3 + 2$
   $= 8$
3. F
4. T
5. F
6. F
7. F
8. F
9. F
10. T
11. F
12. F
13. F
14. T
15. a plus sign

Lesson 3, page 48
1. $2x^2 + 2x + 1$
2. $x^2 + 2x + 7$
3. $2y^2 + 5y + 15$
4. $-y^2 - 2y + x^2 + 5x$
5. $2x^2 + y^2 + xy + 5y + 10$
6. $3x^2 + 2y^2 - 5x + 2xy + 5$
7. $3x^2 + 6x + 5$

Exploration 1, page 48
1. $6x$
2. 22
3. $4x + 2$
4. $2x + 14$
5. $4y + 2$
6. $2y + 2x + 10$

Lesson 4, pages 49-50
1. $-2x$
2. $-5y$
3. $5xy$
4. $-3x^2$
5. cannot simplify
6. $-9x$
7. $x^2 - 8x + 9$
9. $-5x + 4$
10. $x^2 - 8x + 7$
11. $2y^2 + 4y - 10$
12. $-5y - 4$
13. $-7x^2 + 8x - 5$
14. $-3y^2 + 3xy + x + 1$
15. $2x^2 - 2y^2 + x + 2xy - 7$
16. $6x^2 - y^2 - 5xy - 5y + 10$
17. $y^2 + 3x^2 - 4y - 5$
18. $6x + 9$

Lesson 5, page 51
1. b.
2. b.
3. d.
4. d.
5. a.
6. c.
7. $2x^2 - 4 + x + x^2 = 3x^2 + x - 4$
8. $2x^2 - 4 - x + x^2 = 3x^2 - x - 4$

Exploration 2, page 52
1. $4x$; $2x$; impossible to tell
2. $5x + 3 > 5x - 5$
3. $6x < 6x + 2x^2 + 3$
   because $2x^2 + 3$ is positive
4. $2x = 2x$
5. $2x + 3$; $3x + 2$;
   impossible to tell
6. $7x = 7x$

Exploration 3, page 52
1. $(x + 2)(x + 1)$
2. $(2x + 4)(x + 1)$ or
   $(x + 2)(2x + 2)$
3. $(4x + 2)(x + 1)$ or
   $(2x + 2)(2x + 1)$
4. $(x + 1)(x + y)$
5. $(x + 5)(x + 2)$
6. $(3x + 2)(x + 1)$
7. $(3x + 2)(2x + 1)$
8. $(3x + 2)(2x + 5)$
9. $(3x + 1)(x + 5)$
10. $(2x + 2)(2x + 2)$

Exploration 4, page 53
1. $2x^2 + 4x + 4$
2. $2y^2 + 4y + 2x + 2$
3. $16y + 10$
4. $2xy + 2y + 2x + 12$
5. $80 + 4x$

Exploration 5, page 53
1. S ($x = 1$)
2. N
3. S ($y = 1$)
4. A
5. N
6. A

Chapter 5

Exploration 1, page 55
1. $x(x + 7)$
2. $(x + 6)(x + 1)$
3. $(x + 5)(x + 2)$
4. $(x + 4)(x + 3)$
5. $(x + 6)(x + 2)$
6. $(x + 12)(x + 1)$

Exploration 2, page 55
1. $6 \cdot 6$
2. $7 \cdot 7$
3. not possible
4. $(2x)(2x)$
5. $(3x)(3x)$
6. $(x + 1)(x + 1)$
7. $(x + 3)(x + 3)$
8. $(2x + 1)(2x + 1)$
9. not possible
10. not possible
11. $(3x + 2)(3x + 2)$
12. $(x + y)(x + y)$

Lesson 1, page 56
1. $-2x$
2. $3y^2$
3. $-2xy$
4. is positive
5. is negative

Exploration 3, page 56
1. ?
2. N or 0
3. ?
4. P or 0

Lesson 3, page 59
1. $3x(2x+1) = 6x^2 + 3x$
2. $6x^2 - x - 1$
3. $2x^2 - 2x$
4. $y^2 + 4y$
5. $3x^2 + 3xy - 15x$
6. $4xy - 2y^2 + 12y$
7. $6x^2 - 9x$
8. $3x^2 + 13x - 10$
9. $y^2 - y - 12$
10. $2x^2 + 2xy + 8x + 4y + 8$
11. $2xy - 5x + x^2$
12. $2x^2 - x - 6$
13. $y^2 + 2xy + 2y + 10x - 15$
14. $2y^2 + xy - x^2 - 3y - 3x$

Lesson 4, page 60
1. $5x + xy$
2. $5x + xy$
3. $xy + 5y$
4. $xy + 5y$
5. $5x + 5y$
6. $xy - 5x$
7. $5y - xy$
8. $5y - 5x$

10. a. rs + rt
    b. rt − st
11. a. $-x - y$
    b. $x - y$
    c. $-x + y$
12. a. $-x - y$
    b. $x - y$
    c. $-x + y$

Lesson 5, pages 61-62
1. $(6 - x)(3x - 2)$
2. $(6 - x)(3x - 2)$
3. $-3x^2 + 20x - 12$
4. $-2x^2 + 11x - 12$
5. $2y^2 - 11y + 5$
6. $3x^2 + 7x + 2 +$
7. $2y^2 + 16y + 24$
8. $3x^2 - 5x - 2$
9. $2y^2 - 6y - 8$
10. $-2x^2 + 13x - 15$
11. $y^2 - 8y + 15$
12. $3x^2 + 12x - 15$
13. $2x^2 + 15x - 18$
14. $-2y^2 + 14y - 12$
15. $3x^2 + 5x - 2$

Lesson 6, pages 63-64
1. $2x + 1$
2. $(2x^2 + 5x + 4) / (2x + 1)$
3. $x + 2$
4. 2
5. $3x + 2$
6. $x + 2$
7. $3x + 1$
8. $2x + 2$
9. $3x + 1$, remainder 2
10. $x + 1$, remainder 6
11. $2x$, remainder 10
12. $x + 1$, remainder 4
13. $2x + 1$

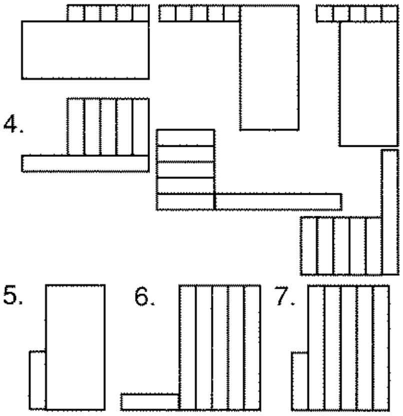
Lesson 7, pages 65-66
1. $(y^2 - 3y + 2) / (y - 2)$
2. $y - 1$
3. $(y^2 + 3y - 10) / (y - 2)$
4. $y + 5$
5. $y - 2$
6. $3x - 3$
7. $x + 3$
8. $y - 2$
9. $x + 6$, remainder 7
10. $3x + 1$
11. $x - 3$
12. $2y + 5$, remainder 15

Lesson 8, page 67
1. 81
2. 3
3. 6
4. 6,7
5. 9,10
6. 7,8
7. 10,11
8. $4x^2$
9. $x^2 + 4x + 4$
10. $9y^2$
11. $y^2 + 6y + 9$

Exploration 4, page 69
1. If x is positive, $2x$ is greater, if x is negative, x is greater.
2. $x^2 > x$ if $x < 0$ or $x > 1$;
   $x^2 < x$ if $0 < x < 1$;
   $x^2 = x$ if $x = 0$ or $x = 1$
3. $x^2 > -1$
4. $x^2 \geq 0$
5. $x^2 + 1 \leq 3x^2 + 1$
6. $4 - 2x - x^2 > -2x^2 - 4x - 4$

Exploration 5, page 69

1. 2. 3. 4. 5. 6. 7.



8. $2y + 2x + 2$
9. $4y + 30$
10. $2y + 6x + 10$

Exploration 6, page 70
1. v = 1, s.a. = 6
2. v = 2, s.a. = 10
3. v = 4, s.a. = 16
4. v = 8, s.a. = 24
5. v = x, s.a. = $4x + 2$
6. v = $2x$, s.a. = $6x + 4$
7. v = $4x$, s.a. = $8x + 8$
8. v = $8x$, s.a. = $16x + 8$
9. v = x^2, s.a = $2x^2 + 4x$
10. v = $2x^2$, s.a. = $2x^2 + 8x$
11. v = $4x^2$, s.a. = $4x^2 + 12x$
12. v = $8x^2$, s.a. = $8x^2 + 16x$

Exploration 7, page 70
1. N
2. S ($x = \frac{1}{4}$)
3. A
4. A
5. A
6. S, when $x = 0$

Chapter 6

Exploration 1, page 72
1. $(x - 1)(x + 1) = x^2 - 1$
2. $(y - 1)(y + 1) = y^2 - 1$
3. $(5 - x)(5 + x) = 25 - x^2$
4. $(y - 5)(y + 5) = y^2 - 25$
5. $(y - x)(y + x) = y^2 - x^2$
6. $(y - 3)(y + 3) = y^2 - 9$
7. $(2x - 2)(2x + 2) = 4x^2 - 4$

Lesson 1, pages 72-73
1. 3 plus 4, times 2, plus 1.
4. $4 \bullet (2 + 3) = 20$
5. $\frac{1}{4} + \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{2}$

6. $5 \cdot (3 - 2 + 6) = 35$
7. $(32 + 2) \cdot (7 - 4) = 33$
8. $\frac{1}{3}(6 + 4 \cdot \frac{2}{6} - \frac{1}{3}) = \frac{7}{3}$
9. $(1 - 2) \cdot (2 + 5) \cdot 6 = -42$
10. $(4 + 6) \cdot (2 \cdot (5 - 3)) = 40$
11. $(3 + 1) \cdot (7 - 22) \cdot (9 - 7) = 24$
12. $2 \cdot 8 \cdot (\frac{1}{4} + \frac{2}{3}) \cdot (2 - \frac{1}{2}) = 22$
13. $\dfrac{5 \cdot \sqrt{12 + 4}}{6 - 2} = 5$
14. $\sqrt{\dfrac{(2 + 6)\,3^2}{2}} = 6$
15. $\sqrt{3 \cdot (2 + 4) \cdot 2^3} = 12$

Lesson 2, page 74
1. $x^2 + xy + 6x + 5y + 5$
2. $x^2 + xy + x + 2y - 2$
3. $-x^2 + 6x + y^2 - 4y - 5$
5. $6x^2 + 5x - 6$
6. $-11x + 2x^2 - 5y + 2xy + 15$
7. $y^2 + y - 12$
8. $xy + y^2 - y - 2x^2 + 13x - 20$
9. $3x^2 + 11x - 20 - 3xy + 4y$
10. $3xy + x - 2x^2 + 2y^2 - 7y + 3$
11. $2x^2 + 3xy + x + y^2 - 1$
12. $4x^2 + 4xy - 8x + y^2 - 4y - 5$
13. $-x^2 + 10x + y^2 - 25$

Exploration 3, page 75
1. P
2. N
3. N
4. ?
5. ?
6. ?
7. 2 and 3
8. 5 and 6

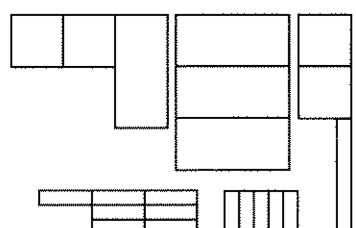
Lesson 4, page 76
1. $x^2 + 6x + 9$
2. $x^2 + 10x + 25$
3. $x^2 + 2xy + y^2$
5. False
6. $y^2 - 4$
7. $y^2 - 25$
8. $y^2 - x^2$
10. $y^2 - 6y + 9$
11. $y^2 - 10y + 25$
12. $y^2 - 2xy + x^2$
14. False
15. $4x^2 - 12x + 9$
16. $4y^2 + 4xy + x^2$
17. $16y^2 - 1$
18. $9x^2 - 16$
19. $9y^2 + 30y + 25$
20. $25x^2 - 20xy + 4y^2$
21. $x^2 + 2xy + y^2 + 4x + 4y + 4$
22. $x^2 + 2xy + y^2 - 10y - 10x + 25$
24. $a^2 - 2ab + b^2 + 2ca - 2cb + c^2$

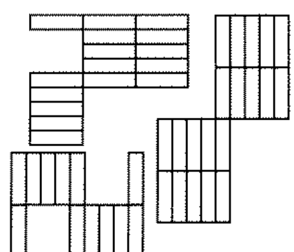
Lesson 5, pages 77-79
1. $(6x^2 + 5x - 10) / (2x + 5)$
2. $(3x - 5)$, remainder 15
3. $2x + 3$
4. $2x$, remainder 5
5. $y + 6$, remainder -30
6. $2x$, remainder 13

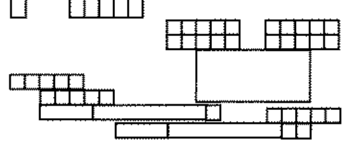
Lesson 6, page 80
1. $2y$, remainder 5
2. $4x - 3$, remainder 5
4. $x^2 - 2x + 3$, remainder -3
5. $2x + 3$, remainder 3

Exploration 4, page 81
1. When x is negative, $x^2 > x$. When $x = 0$ or 1, $x^2 = x$. When $0 < x < 1$, $x^2 < x$. When $x > 1$, $x^2 > x$.
2. When $x = 0$, $x^2 = 2x^2$. For all other values, $x^2 < 2x^2$.
3. When $x > 1$, $x^3 > x^2$. When $x = 0$ or 1, $x^3 = x^2$, otherwise $x^3 < x^2$.
4. When $x = -1$, 0, or 1, $x^3 = x$. If $x < -1$ or $0 < x < 1$, $x^3 < x$. If $-1 < x < 0$ or $x > 1$, $x^3 > x$.
5. $-3x + 13$? $-x + 7$
   The two sides are equal when $x = 3$. When $x < 3$, the left side is greater.
6. $x^2 + 4x - 2 > -x^2 + 4x - 2$

Exploration 5, page 82

1.


2.


3.


Exploration 6, page 82
1. $2y^2 + 8x + 6y + 2$
2. $12y + 2x + 34$

Chapter 7

Exploration 1, page 84
1. The sides are equal if $x = -3$. The left side is greater for $x < -3$.
2. $4x < 4x + 5$
3. The sides are equal if $x = \pm 1$. The left side is greater for $-1 < x < 1$.
4. $8x + 9 > -x^2 + 8x + 9$
5. $3x^2 + 2x > 3x^2 + 2x - 5$
6. The sides are equal if $x = -5$. The left side is greater for $x < -5$.
7. The sides are equal if $x = -13$. The left side is greater for $x > -13$.

Lesson 1, page 84-86
1. N
2. S
3. A
4. $x = 5$
5. $2x + 22 - 1 - (x + 5 + 1) = 6x + 11 - 1 - (x + 5 - x)$
6. $x = 2$
8. $x = 3$
9. $3 = x$
10. $x = 5$
11. $x = -1$
12. $x = -1$
13. $x = \frac{7}{5}$

Exploration 2, page 88
1. $x = -1$
2. $x = \frac{15}{2}$
3. $x = \pm 4$
4. identity
5. $x = 0$
6. identity
7. $x = 0$
8. $x = 0, \pm 1$
9. no solutions with real numbers
10. $x = -2$
11. $x = -2, 3$
12. $x = \pm\sqrt{10}$

Lesson 2, page 88
1. $y = 14$
2. $x = 3$
3. $x = 5$
4. $x = 2$
5. $x = 8$
6. $x = 2$

Lesson 3, pages 89-91

1. $3x + 13 = 6x - 5$
2. $x = 6$
3. $x = -3$
4. $x = 3$
5. $x = -2\frac{1}{2}$
6. $x = 4\frac{1}{2}$
7. $x = 2$
8. $x = -\frac{5}{3}$
9. $x = -\frac{23}{2}$
10. $x = \frac{12}{5}$

Lesson 4, page 92

3. $x = -4$
4. $x = \pm 5$
5. $x = 1\frac{1}{2}$
6. identity
7. $x = -2\frac{1}{2}$
8. no solutions
9. $x = -\frac{25}{3}$

Lesson 5, page 94

1. a, c, one solution for e
2. a, b, c, e (two solutions)
3. a, b, c, d, e

Exploration 3, page 95

1. $x(x + 10) = x^2 + 10x$
   $(x + 1)(x + 9) = x^2 + 10x + 9$
   $(x + 2)(x + 8) = x^2 + 10x + 16$
   $(x + 3)(x + 7) = x^2 + 10x + 21$
   $(x + 4)(x + 6) = x^2 + 10x + 24$
   $(x + 5)(x + 5) = x^2 + 10x + 25$
2. $(x + 1)(x + 18) = x^2 + 19x + 18$
   $(x + 2)(x + 9) = x^2 + 11x + 18$
   $(x + 3)(x + 6) = x^2 + 9x + 18$

Exploration 4, page 95

1. $(x + 5)(x + 5) = x^2 + 10x + 25$
2. $(2x + 2)(2x + 2) = 4x^2 + 8x + 4$
3. $(3x + 1)(3x + 1) = 9x^2 + 6x + 1$
4. $(x + 1)(x + 1) = x^2 + 2x + 1$
5. $(2x + 3)(2x + 3) =$
   $4x^2 + 12x + 9$
6. No
7. $(x + 5)(x + 5) = x^2 + 10x + 25$
8. $(2x + 5)(2x + 5) =$
   $4x^2 = 20x + 25$
9. $(x + 6)(x + 6) = x^2 + 12x + 36$
10. $(3x + 1)(3x + 1) = 9x^2 + 6x + 1$
11. $(x + 3)(x + 3) = x^2 + 6x + 9$
12. No

Exploration 5, page 96

1.

1	4
2	6
3	8
4	10
10	22
100	202
n	$2n + 2$

2.

1	12
2	22
3	32
4	42
10	102
100	1002
n	$10n + 2$

3.

1	12
2	14
3	16
4	18
10	30
100	210
n	$2n + 10$

4.

1	12
2	16
3	20
4	24
10	48
100	408
n	$4n + 8$

5.

1	12
2	20
3	28
4	36
10	84
100	804
n	$8n + 4$

6.

1	12
2	18
3	24
4	30
10	66
100	606
n	$6n + 6$

Exploration 6, page 96

1.

1	6
2	10
3	14
4	18
10	42
100	402
n	$4n + 2$

2.

1	22
2	34
3	46
4	58
10	130
100	1210
n	$12n + 10$

3.

1	22
2	36
3	50
4	64
10	148
100	1408
n	$14n + 8$

4.

1	22
2	38
3	54
4	70
10	166
100	1606
n	$16n + 6$

Chapter 8

Exploration 1, page 98

1. T
2. T
3. T
4. F
5. If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.
6. Either $a = 0$ or $b = 0$.

Lesson 1, pages 98-99

2. $x = -6$, or $x = \frac{1}{2}$
3. $x = -6$, or $x = \frac{1}{2}$
4. $x = -\frac{1}{3}$, or $x = -5$
5. $x = 5$, or $x = -\frac{3}{2}$
6. $x = 1$, or $x = \frac{1}{3}$

Lesson 2, page 100

4. $(2x^2 + 11x - 6) / (x^2 + 6x)$
5. $(2x - 1) / x$
6. $2x / (x + 1)$
7. $4x / (x - 1)$
8. $(2x-1) / 3$

Lesson 3, pages 102-103

1. $xy + x^2 + 3x$
2. $x(y + x + 3)$
3. $x(2x - 1)$
4. not possible
5. $x(x + 2 + y)$
6. $3x(x - 1)$
7. $(x + 3)^2$
8. $(y - x)^2$
9. $(y + 2)(y - 2)$

Exploration 2, page 103

1. $x > -4$
2. $x > 6$
3. $x > -6$
4. $x > 6$
5. $x > 6$
6. $x > 4$
7. $x > 4$
8. $x > -6$
9. $x < -6$
11. $x < 6$

Lesson 4, pages 104-105

1. $(x + 2)^2$
2. $(3x + 1)^2$
3. not possible
4. not possible
5. $(x + y)^2$
6. not possible
8. $y^2 - 4y + 4$
9. $(y - 2)^2$
10. $(2y - 1)^2$
11. not possible
12. $(y - 3)^2$
13. $(y - 5)^2$
14. The square of a difference: $(a - b)^2 = a^2 - 2ab + b^2$
15. $y^2 - 25$
16. $(y + 5)(y - 5)$
17. $(x + 1)(x - 1)$
18. $(y + 2)(y - 2)$
19. $(y + x)(y - x)$
21. $(2x + 1)^2$
22. $x(x + 2)$
23. $(y + 3)(y - 3)$
24. $(y - 4)^2$
25. not possible
26. $y = 4$ or -4
27. $x = -2$
28. $x = 0$ or -3
29. $x = -\frac{3}{2}$
30. $y = 4$

Lesson 5, page 106

1. $6x^2 + 8x + 2$
2. $(2x + 2)(3x + 1)$
3. $(3x + 1)^2$
4. $(2x + 1)(x + 2)$
5. $(x + 2)(x + 5)$
6. not possible
7. $(2x + 1)(x + 5)$

Lesson 6, page 107

1. $6x^2 - 8x + 2$
2. $(3x - 1)(2x - 2)$
3. $(y - 5)(y - 1)$
4. $(y - x)(y - 2)$
5. $(2x - 5)(2x - 1)$
6. $(3x - 2)(x - 1)$

Lesson 7, page 108

2. $6x^2 + x - 2$
3. $(2x - 1)(3x + 2)$
4. $y^2 - 2y - 3$
5. $2x^2 + 8x - 10$
6. $(3x - 5)(x + 2)$
7. $(x + 3)(2x - 4)$
8. $(2x - 3)(2x + 1)$
9. $(y - x)(y + 5)$

Lesson 8, page 109

1. $y(x + 6 + y)$
2. $(y + 4)(y - 4)$
3. $(3x - 2)(x + 5)$
4. $(2x + 2)(2x + 2)$
5. impossible
6. $(y - 2)(y - 3)$
7. $(y - 2)(y - 2)$
8. $(x + 6)(x + 2)$

Exploration 2, page 109

1.

1	$2x + 2$
2	$4x + 2$
3	$6x + 2$
4	$8x + 2$
10	$20x + 2$
100	$200x + 2$
n	$2nx + 2$

2.

1	$2x + 2$
2	$2x + 4$
3	$2x + 6$
4	$2x + 8$
10	$2x + 20$
100	$2x + 200$
n	$2x + 2n$

3.

1	$2x + 10$
2	$2x + 20$
3	$2x + 30$
4	$2x + 40$
10	$2x + 100$
100	$2x + 1000$
n	$2x + 10n$

4.

1	$2x + 10$
2	$4x + 10$
3	$6x + 10$
4	$8x + 10$
10	$20x + 10$
100	$200x + 10$
n	$2nx + 10$

5.

1	$2x + 10$
2	$4x + 12$
3	$6x + 14$
4	$8x + 16$
10	$20x + 28$
100	$200x + 208$
n	$2nx + 2n + 8$

Lesson 9, page 110

1. $(x^2 + 7x + 10) / (xy + 2y - x^2 - 2x)$
2. $(x + 5)(x + 2) / (y - x)(x + 2)$
3. $(x + 5) / (y - x)$
4. $5 / (y + x)$
5. $(x + 1) / (x - 2)$
6. $(x + 5) / (2x + y)$
7. $(y - 4) / (y - 3)$
8. impossible

Exploration 3, page 111

1.

1	$4x + 2$
2	$8x + 2$
3	$12x + 2$
4	$16x + 2$
10	$40x + 2$
100	$400x + 2$
n	$4nx + 2$

2.

1	$4x + 2$
2	$6x + 4$
3	$8x + 6$
4	$10x + 8$
10	$22x + 20$
100	$202x + 200$
n	$(2n + 2)x + 2n$

3.

1	$12x + 10$
2	$24x + 10$
3	$36x + 10$
4	$48x + 10$
10	$120x + 10$
100	$1200x + 10$
n	$12nx + 10$

4.

1	$12x + 10$
2	$24x + 14$
3	$36x + 18$
4	$48x + 22$
10	$120x + 46$
100	$1200x + 406$
n	$12nx + 4n + 6$

Chapter 9

Exploration 1, page 113

1. $y = 6 + 4x$
2. $y = 5 - 2x$
3. $y = 4 + 6x$
4. $y = 3 + 2x$
5. $y = 2x - 4$
6. $y = -x/2 + 4$
7. $y = x - 1$
8. $y = \frac{6}{5}x$

Exploration 2, page 113

1. b. $x = 4$, $y = 1$
2. $x = 4$, $y = 6$
3. $x = -\frac{7}{3}$, $y = \frac{11}{3}$
4. $x = \frac{3}{2}$, $y = 3$
5. $x = 8$, $y = -2$
6. $x = \frac{3}{2}$, $y = \frac{1}{2}$
7. $x = \frac{5}{3}$, $y = \frac{1}{3}$
8. either $x = 0$, $y = 0$, or $x = 1$, $y = 1$
9. $x = 2$, $y = \frac{1}{2}$
10. $x = -4$, $y = -10$
11. $x = 3$, $y = 1$
12. either $x = 2$, $y = 1$, or $x = -1$, $y = -\frac{1}{2}$

Exploration 3, page 114

1. $x = -2$, $y = -6$
2. $x = 3$, $y = -2$
3. $x = 11$, $y = -3$
4. $x = 2$, $y = 1$
5. $x = 2$, $y = -\frac{1}{5}$
6. $x = -3$, $y = 0$
7. $x = -4$, $y = 3$
8. $x = 0$, $y = 3$
9. $x = 3$, $y = 7$
10. $x = -3$, $y = 5$
11. $x = -1$, $y = 5$
12. $x = 4$, $y = 1$

Lesson 1, pages 114-117

1. a. 11
   b. 9
   c. 15
   d. 10
   e. an infinite number
3. $x + 2y = 11$, $x - 2y = 3$
4. $x = 2y + 3$
5. $y = 2$
6. $x = 7$
8. $x = 4$, $y = 1$
9. $x = -3$, $y = 5$
10. All values that satisfy one equation satisfy the other.
11. $x = -2$, $y = -3$
12. no solution
13. $x = -1$, $y = 3$

Lesson 2, page 118

1. $x = -\frac{2}{3}$
2. $x = -2$ or $x = -\frac{5}{2}$
3. $x = \frac{3}{2}$ or $x = -\frac{3}{2}$
4. $x = -\frac{1}{3}$ or 1
5. $x = 0$ or $-\frac{5}{2}$
6. cannot be factored
7. $x = 1$
8. $x = \frac{5}{3}$ or $\frac{2}{3}$

Lesson 3, pages 118-119

1. $x = 5$ or -5
3. $x^2 + 10x + 25 = 49$
4. $x = 2$ or $x = -12$
5. $x = 4$ or -4
6. $x = -1$
7. $x = 3$ or -3
8. no real solution
9. $x = -1$ or 2
10. $y = \sqrt{5} - 5$
12. $x = 2$ or -8

Lesson 4, pages 120-121

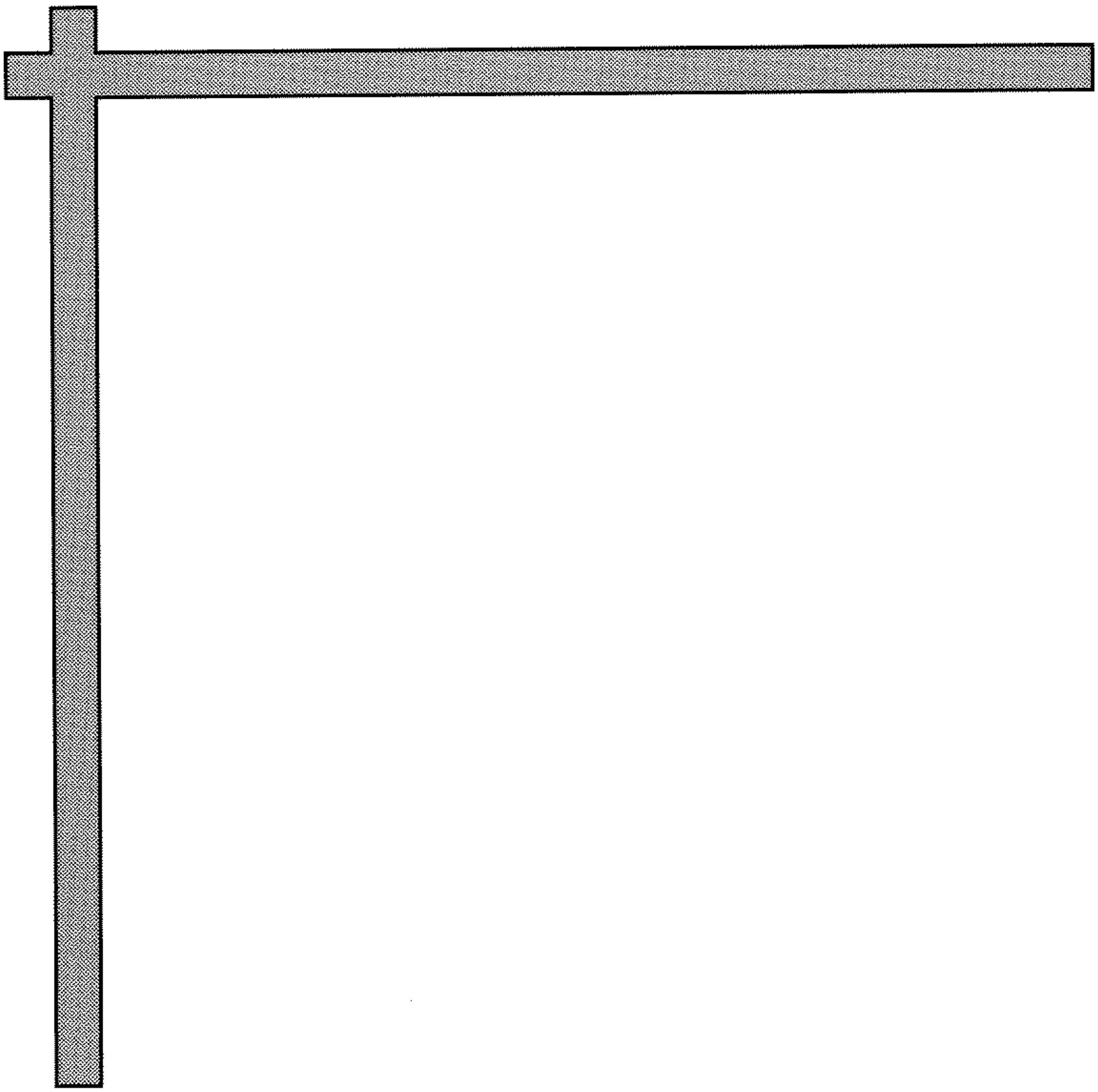
1. $x^2 + 6x = -5$
2. 9
3. $x = -1$ or -5
4. $x = 1$ or -11
5. $y = 6$ or -4
6. $y = 6$ or -2
7. $y = 3$
8. $y = 7$ or 1
9. $y = -7$ or -1
10. $x = -3$

Exploration 4, page 122

1	1	4
2	3	8
3	6	12
4	10	16
10	55	40
100	5050	400
n	$\frac{n(n+1)}{2}$	$4n$

Exploration 5, page 122

1	1	6
2	4	18
3	10	36
4	20	60
10	220	330
100	171700	30,300
n	$\frac{n(n+1)(n+2)}{6}$	$3n(n+1)$



Exploration 1 Solving for y

Set up these problems with the Lab Gear, and rearrange them so that y is by itself on the left. Write equations to show all the steps. In some cases, you will need to finish the problem without the blocks.

1. $y - 4x = 6$
2. $2y + 4x - 10 = 0$
3. $-6x + y = 4$
4. $3y - 6x = 9$
5. $6x - 3y = 12$
6. $2y + x = 8$
7. $x - y = 1$
8. $6x - 5y = 0$

Exploration 2 Solving for x and y

1. a. Set up this equation with the blocks: $2x + y = 9$.
   Using trial and error, find some values of x and y that make it true.
   b. Suppose you are told that x is 3 more than y. Were any of your answers to problem 1 correct? If not, can you solve the equation correctly? (Hint: Set up the equation with only one variable.)

For each of these problems, set up the equation with the blocks, then use the extra fact to solve the equation. Write the value of x and y.

2.	$4x - 7 = y + 3$	y is 2 more than x
3.	$2y + x = 5$	x is 6 less than y
4.	$3y + 2x = 12$	y is twice x
5.	$2y + x = 4$	x and y add up to 6
6.	$4x - 2y = 5$	y is one-third of x
7.	$2x = 6 + y$	x divided by y equals 5
8.	$x^2 - 2x + y = 0$	y is the square of x

Try solving these equations without the Lab Gear.

9.	$y + \frac{3}{x} = 2$	x is the reciprocal of y
10.	$\frac{1}{x-1} = \frac{2}{y}$	y is 2 more than 3 times x
11.	$6 - \frac{3}{x-2} = \frac{3}{y}$	y is 2 less than x
12.	$\frac{x-1}{2} = \frac{1}{2y}$	x is twice y (Hint: there is more than one possible answer.)

Exploration 3 More Solving With Two Variables

Use the Lab Gear to set up the first equation. Then, use the extra fact to substitute other blocks for the x-blocks or the y-blocks.

1. $2x - y = 2$	$y = 3x$
2. $4x + y = 10$	$y = 6x - 20$
3. $x - 4y = 23$	$x = -5y - 4$
4. $3y + 2x = 7$	$3y = 4x - 5$
5. $5y - 4x = -9$	$5y = 3x - 7$

For the next problems, you may need to rewrite the second equation for it to be helpful.

6. $5x + 3y = -15$	$y = 2x + 6$
7. $5x - 3y = -29$	$x = 2 - 2y$
8. $2x + 3y = 9$	$4x = 6 - 2y$
9. $4x - y = 5$	$3y = 6x + 3$
10. $x - 3y = -18$	$2x + 3y = 9$
11. $6x - 2y = -16$	$4x + y = 1$
12. $3x + 5y = 17$	$2x + 3y = 11$

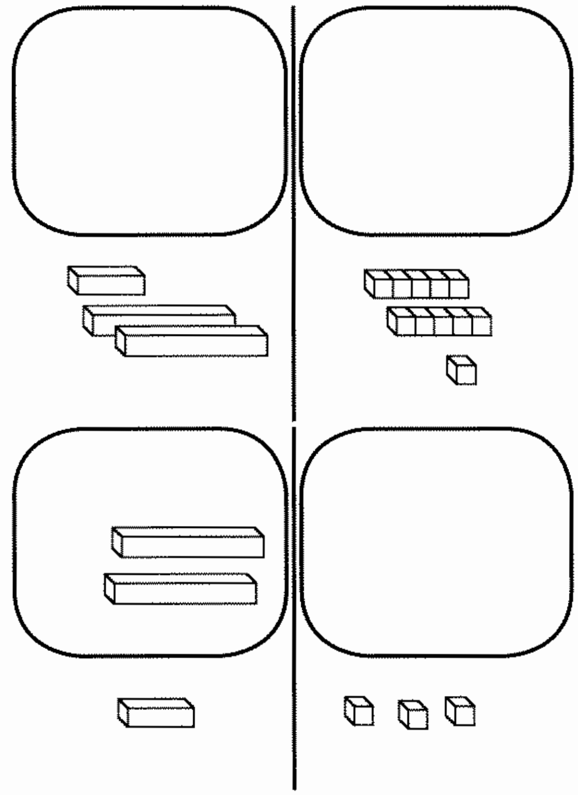
Lesson 1

Simultaneous Equations

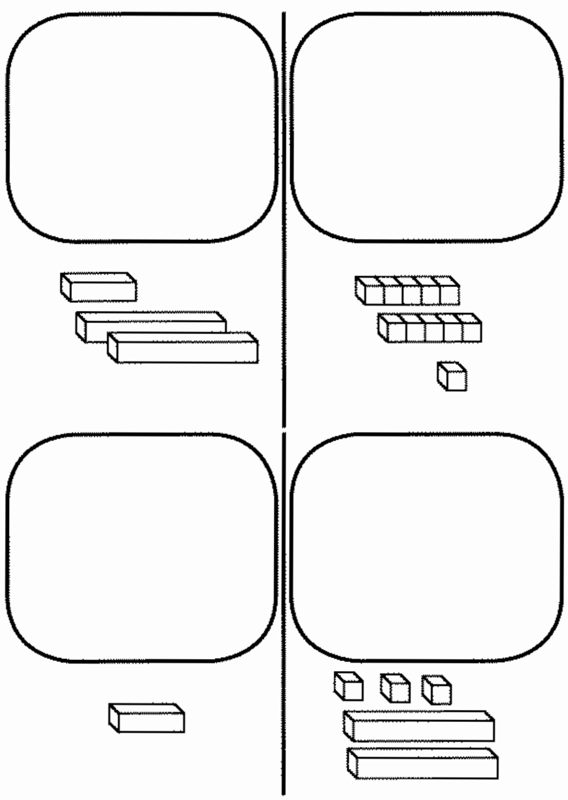
1. Look at this equation: $x + 2y = 11$. Try solving for x.
   Use the Lab Gear, pencil and paper, or mental arithmetic to answer these questions.
   a. If $y = 0$, then $x =$ _____?
   b. If $y = 1$, then $x =$ _____?
   c. If $y = -2$, then $x =$ _____?
   d. If $y = 0.5$, then $x =$ _____?
   e. Write the number of solutions you think there are for this equation.
2. Find four sets of values for x and y that are solutions to the equation $x - 2y = 3$. Try each of them to see whether it is also a solution to the equation in problem 1.

These figures show how to use the Lab Gear to find a set of values for x and y that satisfy two equations *at the same time.*

- You will need two workmats. Show one equation on each workmat.



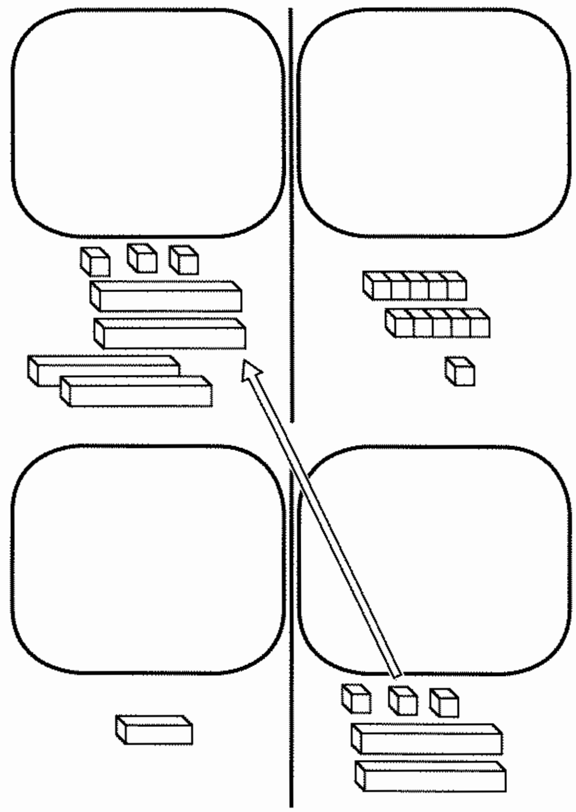
3. Write the two equations.

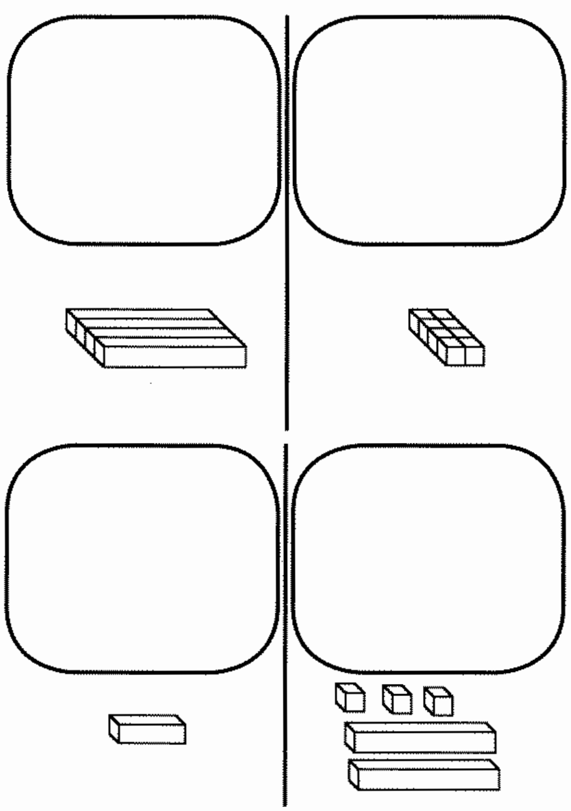


- Next, on one of the workmats, solve for one of the variables in terms of the other. In this case, the bottom workmat is used, and x is evaluated in terms of y.

4. Write the value of x in terms of y, for the bottom workmat.

- Next, x is replaced by $2y + 3$ on the other workmat. (Notice that additional blocks are used. Do not simply slide blocks from one workmat to the other—you will soon see why.)



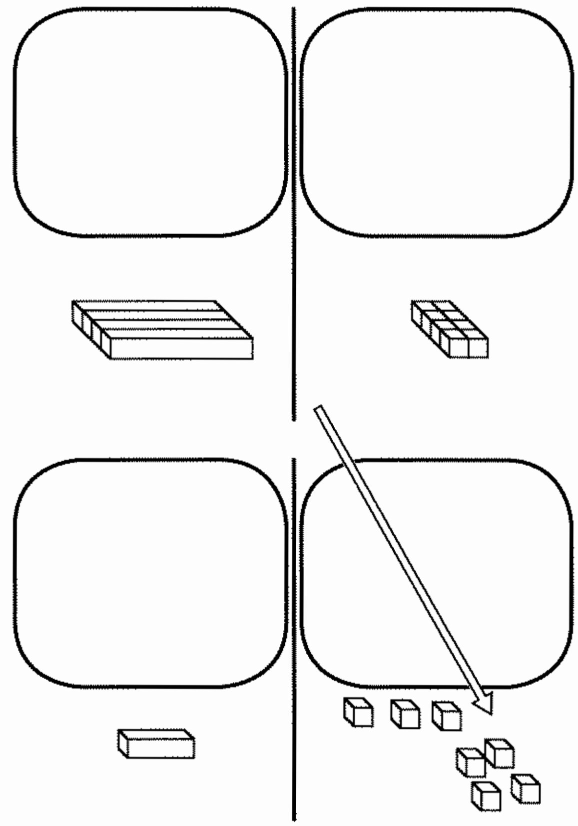


- Now solve for y on the upper workmat. This is straightforward, as there are no x-blocks left to get in the way.

5. Write the value of y.

- Finally, substitute the value you found for y into the lower workmat, and solve the equation there. Again, this is easy to solve, since there are no y-blocks left.

6. Write the value of x.
7. Check that the values you found for x and y are indeed solutions to both original equations.



Solve these systems of **simultaneous equations** with the Lab Gear. Some systems will have a single common solution, as in the example. One will have no common solution. One will be such that whatever numbers work for one equation, also work for the other.

8. $3x + 8y = 20$
   $3x + y = 13$
9. $2x + y = -1$
   $x - 3y = -18$
10. $6x - 2y = 12$
    $y = 3x - 6$
11. $x - y = 1$
    $x + 3y = -11$
12. $4x - y = 2$
    $y = 4x + 1$
13. $y = 7x + 10$
    $y = 4 + x$

☑ Self-check

8. $y = 1$, $x = 4$
12. No solution

Lesson 2

Zero Products

Solve these equations, using factoring and the Zero Product Principle. One cannot be factored.

1. $9x^2 + 12x + 4 = 0$
2. $2x^2 + 9x + 10 = 0$
3. $4x^2 - 9 = 0$
4. $6x^2 - 4x - 2 = 0$
5. $4x^2 + 10x = 0$
6. $2x^2 + 10x + 5 = 0$
7. $x^2 - 2x + 1 = 0$
8. $9x^2 - 21x + 10 = 0$

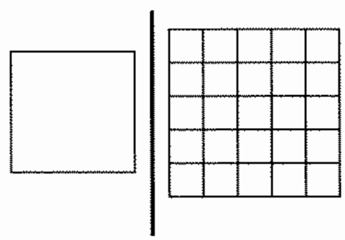
☑ Self-check

8. $x = \frac{2}{3}$ or $x = \frac{5}{3}$

Lesson 3

Equal Squares

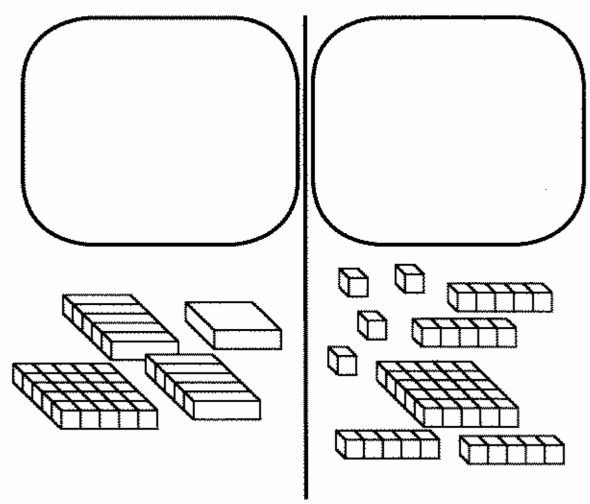
Consider the equation $x^2 = 25$. Put out your blocks like this to illustrate it.



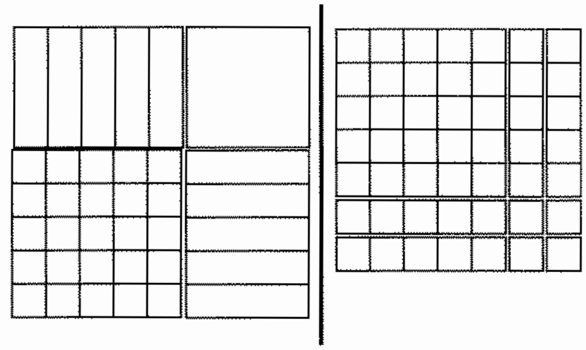
One way to solve this equation is to remember that if the squares are equal, *their sides must be equal.* (This is true even though they don't *look* equal. Remember x can have any value.)

1. Solve the equation. If you only found one solution, think some more, because there are two.
2. Explain why there are two solutions.

3. Write the equation shown by this figure.



By rearranging the blocks, you can see that this is an *equal squares* problem, so it can be solved the same way.



As you can see in the figure, $(x + 5)^2 = 7^2$. It follows that $x + 5 = 7$, or $x + 5 = -7$.

4. Solve the equation. There are two solutions. Check them both in the original equation.

Solve the following equations using the equal squares method. Most, but not all, have two solutions (in one case the solutions are not whole numbers).

5. $x^2 = 16$
6. $x^2 + 2x + 1 = 0$
7. $4x^2 = 36$
8. $4x^2 - 4x + 1 = -9$
9. $4x^2 - 4x + 1 = 9$
10. $y^2 - 10y + 25 = 5$
11. Explain why some problems had one, or no solution.

To solve this problem, remember that if quantities are equal, *their opposites must be equal.*

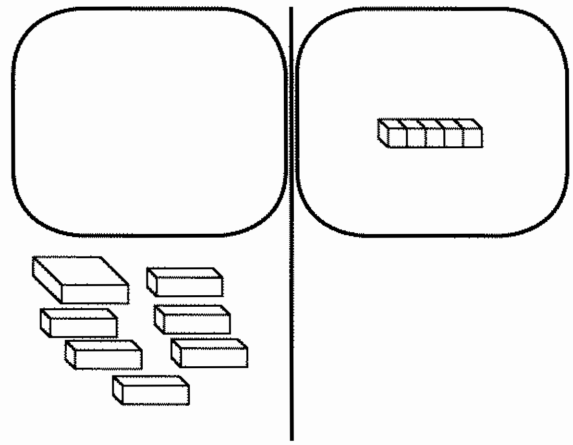
12. $-x^2 - 6x - 9 = -25$

☑ Self-check

9. $x = 2$, or $x = -1$

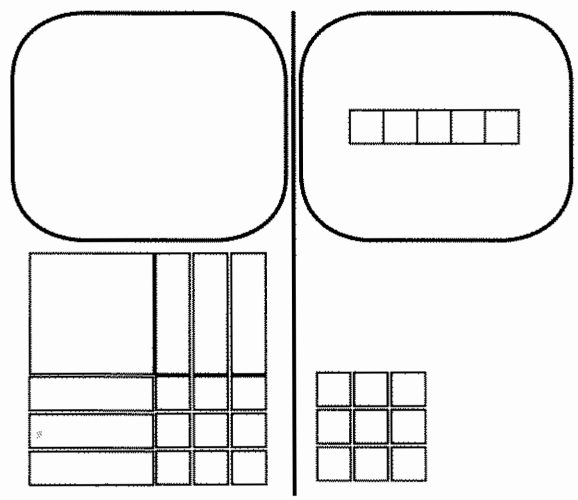
Completing the Square

1. Write the equation shown by this figure. Put out your blocks like this to illustrate it.



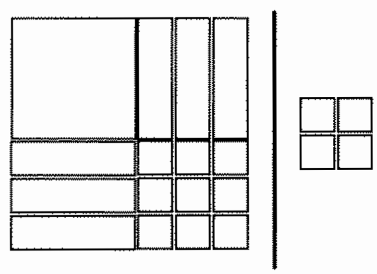
By adding the same quantity to both sides, it is possible to change this equation into an *equal squares* problem. This is called **completing the square**.

2. Write the quantity that was added to complete the square on the left.



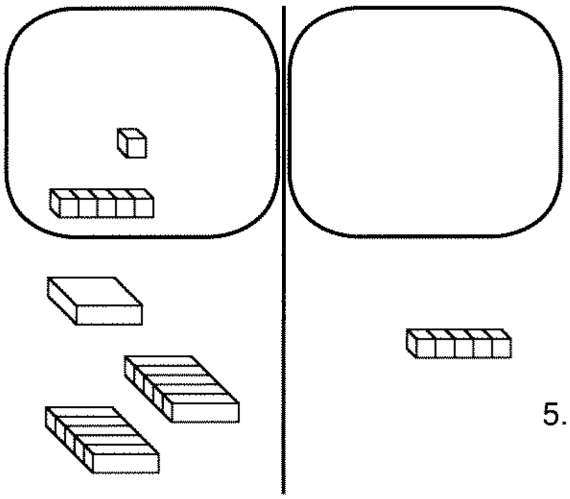
The figure shows what is left after cancelling on the right.

3. Solve the equation. There will be two solutions.

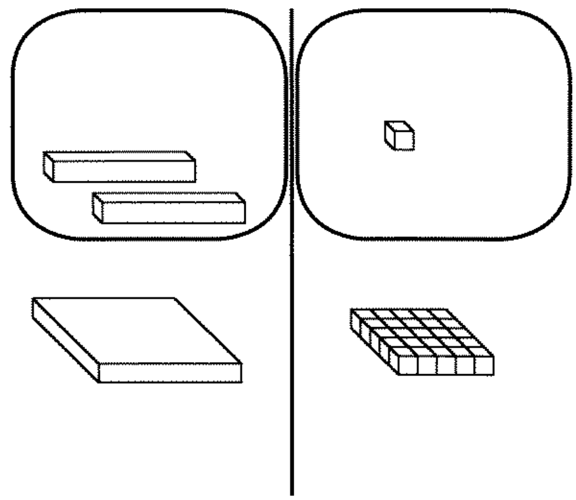


For the following equations, you will need to rearrange the blocks, and add or subtract the same amount on both sides, in order to get equal squares.

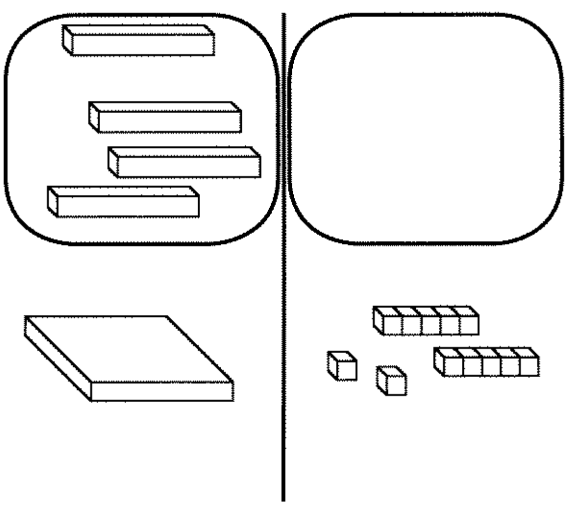
4.



5.



6.



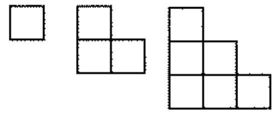
7. $y^2 - 6y = -9$
8. $y^2 = 8y - 7$
9. $-y^2 = 8y + 7$
10. $x^2 + 6x = -9$

☑ Self-check

9. $y = -1$, or $y = -7$

Exploration 4 Area and Perimeter

Look at this sequence. Think about how it continues, following the pattern. Write the perimeters of the figures given, then the perimeter of the fourth one, the tenth one, and the hundredth one. Explain how you got your answers.



Exploration 5 Volume and Surface Area

Look at this sequence. Think about how it continues, following the pattern. Write the volume and surface area of each figure given, then the volume and surface area of the fourth one, the tenth one, and the hundredth one. Explain how you got your answers.