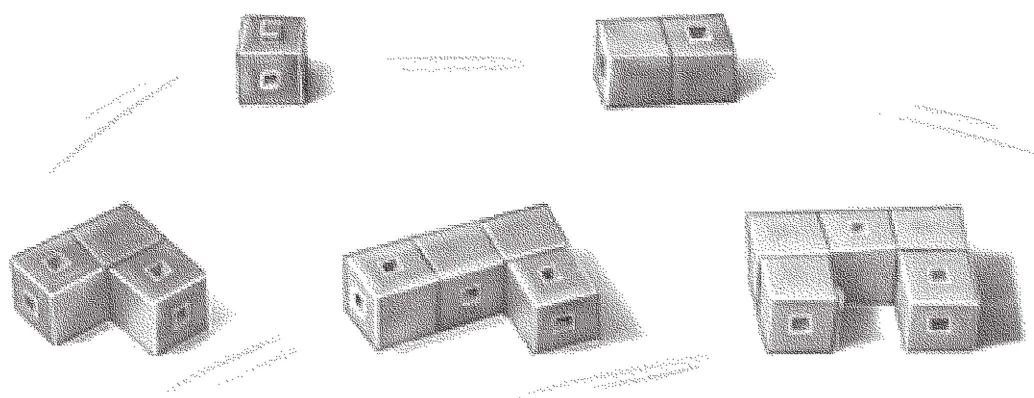
4 Polyominoes



Polyominoes are a major topic in recreational mathematics and in the field of geometric combinatorics. Mathematician Solomon W. Golomb named and first studied them in 1953. He published a book about them in 1965, with a revised edition in 1994 *(Polyominoes: Puzzles, Patterns, Problems, and Packings)*. Martin Gardner's "Mathematical Games" column in *Scientific American* popularized many polyomino puzzles and problems. Since then, polyominoes have become one of the most popular branches of recreational mathematics. Mathematicians have created and solved hundreds of polyomino problems and have proved others to be unsolvable. Computer programmers have used computers to solve some of the tougher puzzles.

Polyominoes have connections with various themes in geometry—symmetry, tiling, perimeter, and area—that we will return to in future sections. The only part of this section that is required in order to pursue those connections is the introductory lab, Lab 4.1 (Finding the Polyominoes). The remaining labs are not directly related to topics in the traditional curriculum, but they help develop students' visual sense and mathematical habits of mind, particularly regarding such skills as:

- systematic searching;
- classification; and
- construction of a convincing argument.

These habits are the main payoff of these labs, as specific information about polyominoes is not important to students' further studies.

Use this section as a resource for students to do individual or group projects, or just as an interesting area for mathematical thinking if your students are enthusiastic about polyominoes.

In the world of video games, tetrominoes are well known as the elements of the game *Tetris,* which is probably familiar to many of your students. In fact, the figures in that game are *one-sided* tetrominoes (see Discussion Question C in Lab 4.2 (Polyominoes and Symmetry)).

Pentominoes (polyominoes of area 5) have enjoyed the greatest success among recreational mathematicians, gamers, and puzzle buffs. A commercial version of a pentomino puzzle called *Hexed* can often be found in toy stores. My book *Pentomino Activities, Lessons, and Puzzles,* available from Creative Publications along with plastic pentominoes, helped bring pentominoes into the classroom.

As a sequel to this work, see Lab 8.1 (Polyomino Perimeter and Area).

See page 184 for teacher notes to this section.

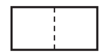
LAB 4.1

Name(s) ______________________

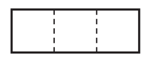
Finding the Polyominoes

■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

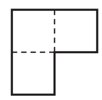
This is a *domino*. It is made of two squares, joined edge to edge.



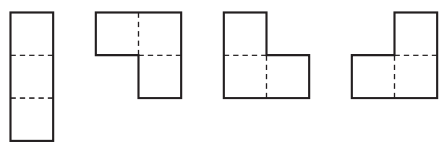
A *tromino* is made of three squares. This one is called the straight tromino.



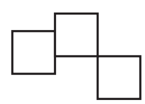
This is the *bent* tromino.



There are only two different trominoes. These are the same ones as above, but in different positions.



However, this is not a tromino, since its squares are not joined edge to edge.



Definition: Shapes that are made of squares joined edge to edge are called *polyominoes*.

You can make polyominoes using interlocking cubes. Be sure that when the figure is laid flat, all the cubes touch the table.

1. *Tetrominoes* are made of four squares. Find them all and record them on grid paper.
2. *Pentominoes* are made of five squares. Find them all and record them on grid paper.

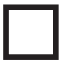
Discussion

A. Find a way to convince an interested person that you have indeed found all of the polyominoes with area from 1 to 5, and that you have no duplicates.

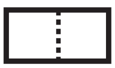
B. A natural way to classify the polyominoes is by area. Find other ways to classify them.

C. Which pentominoes can be folded into a box without a top?

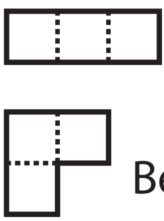
These are the standard polyomino names.



Monomino

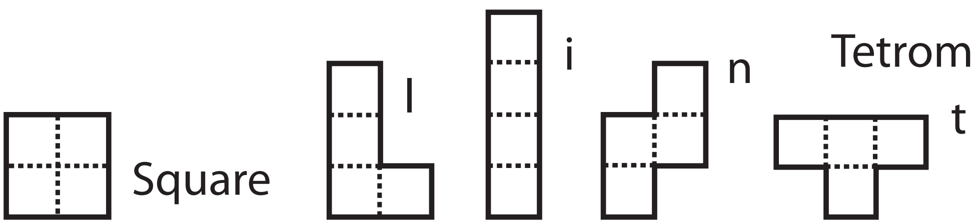


Domino

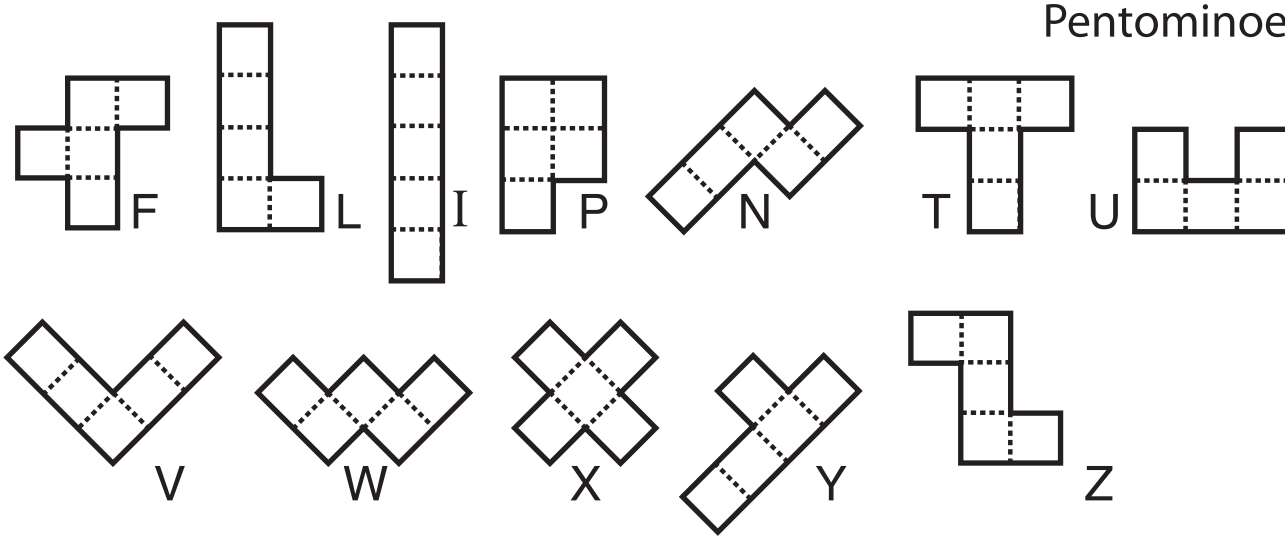


Straight
Bent
Trominoes

Tetrominoes



Pentominoes



LAB 4.2

Name(s) ____________________

Polyominoes and Symmetry

- **Equipment:** 1-Centimeter Grid Paper, interlocking cubes, Polyomino Names Reference Sheet

The straight tromino can be placed on graph paper in two different ways:

1. For each of the polyominoes with area from 1 to 5, how many different positions are there on graph paper?

   Monomino: ______ Domino: ______

   Straight tromino: ______ Bent tromino: ______

   Square: ______ l: ______ i: ______ n: ______ t: ______

   F: ______ L: ______ I: ______ P: ______ N: ______ T: ______

   U: ______ V: ______ W: ______ X: ______ Y: ______ Z: ______

Trace the straight tromino. You can rotate it 180° and it will still fit on its outline. We say that it has two-fold *turn symmetry* or *rotation symmetry*. Alternatively, you can flip it around a vertical axis, or a horizontal one, and it will still fit. We say that it has two *lines of symmetry,* or *mirror lines*.

2. Write the name of each polyomino with area 1 to 5 in the appropriate space in the table below. Some spaces may be empty; others may have more than one entry. For example, the straight tromino is *not* the only polyomino with two mirror lines and two-fold turn symmetry.

	No mirror lines	**1 mirror line**	**2 mirror lines**	**3 mirror lines**	**4 mirror lines**
No turns (except 360°)					
Two-fold turn (180°)			straight tromino		
Three-fold turn (120°)					
Four-fold turn (90°)					

Discussion

A. Which types of symmetry in Problem 2 have no polyomino examples? Why?

B. How are the answers to Problem 1 and Problem 2 related?

C. Imagine that polyominoes were one-sided (like tiles) and that you could not flip them over. Then there would have to be, for example, separate pentominoes for the shapes L and ⅃. In other words, there would be more polyominoes. Find all the additional polyominoes you would need in order to have a full set for areas 1 to 5.

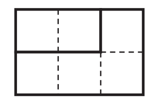
LAB 4.3

Name(s) ____________________

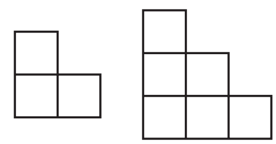
Polyomino Puzzles

■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

1. Using interlocking cubes, make a set of all of the polyominoes with area 2, 3, and 4. You should have eight pieces. It is best to make each polyomino a single color.
2. On graph paper, draw all the rectangles (including squares) that satisfy these two conditions.
   - They have area 28 or less.
   - Their dimensions are whole numbers greater than 1.
3. Use the polyominoes you made in Problem 1 to cover the rectangles you drew in Problem 2. The 2×3 rectangle, covered by a domino and **l** tetromino, is shown in the figure at right. Record your solutions. **Note:** One rectangle is impossible with this set of polyominoes.



4. On Centimeter Grid Paper, draw staircases like the ones in the figures at right, with area between 3 and 28, inclusive. Cover each one with some of the polyominoes you made in Problem 1, and record your solutions.



Discussion

A. Describe your system for finding all the rectangles in Problem 2.

B. Which rectangle puzzle *cannot* be solved in Problem 3? Why?

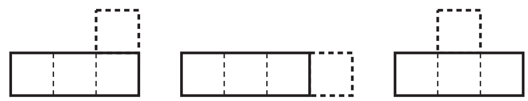
C. Find a formula for the area of a staircase like the two-step and three-step examples in Problem 4, given that it has x steps.

LAB 4.4
Family Trees

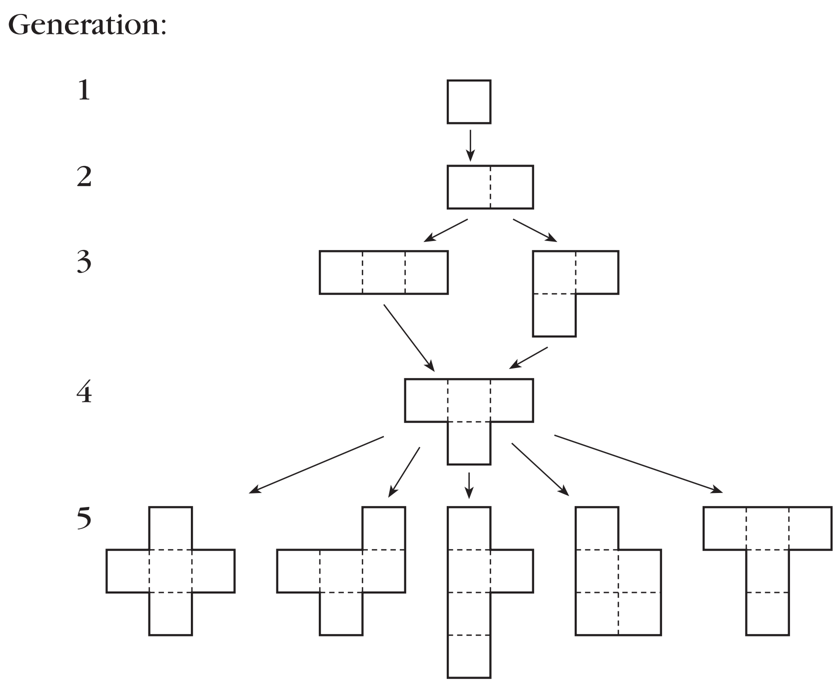
Name(s) ______________________

■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

A polyomino is a *child* of another polyomino if it comes from the original polyomino by the addition of a single square. For example, the **l**, **i**, and **t** tetrominoes are children of the straight tromino. The square and **n** tetrominoes are not.



Here is a family tree for the **t** tetromino. It shows all of its ancestors back to the monomino, and all of its pentomino children.



On grid paper, make complete family trees for the remaining tetrominoes.

1. **l**

2. Square

3. **i**

4. **n**

Name(s) ____________________

Discussion

A. Which pentomino has the most (tetromino) parents?

Two polyominoes "of the same generation" are called siblings (brothers and sisters) if they have a parent in common. For example, the F and the P pentominoes are siblings; both are children of the t tetromino.

B. List all the siblings of the I pentomino.

C. List all the siblings of the W pentomino.

Polyominoes of the same generation are cousins if they are *not* siblings. For example, the square and the straight tetromino are cousins because they do not have a parent in common.

D. List all the cousins of the Y pentomino.

E. Find two "second cousin" pentominoes. (These are pentominoes that have no tetromino or tromino ancestors in common.)

F. Which pentomino has the most (hexomino) children?

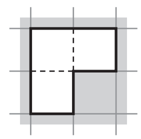
G. Which pentomino has the fewest (hexomino) children?

LAB 4.5
Envelopes

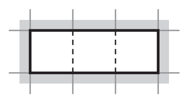
Name(s) ____________________

■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

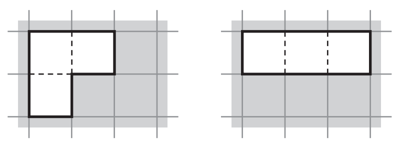
What is the smallest rectangle (or square) onto which you can fit the bent tromino? As you can see in this figure, it is a 2 × 2 square.



A 1 × 3 rectangle is the smallest rectangle onto which you can fit the straight tromino. We say that the 1 × 3 and 2 × 2 rectangles are tromino *envelopes*.

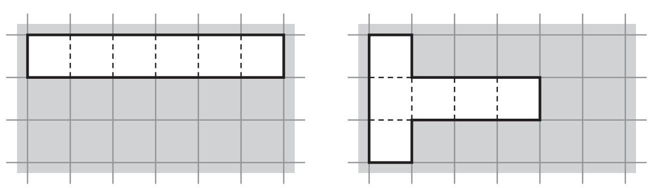


The 2 × 3 rectangle is not a tromino envelope, because it is too big.

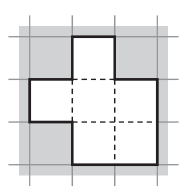


1. There are three tetromino envelopes. Find them and sketch them on grid paper. Which tetrominoes belong with each?
2. There are four pentomino envelopes. Find them and sketch them on grid paper. Which pentominoes belong with each?

The 3 × 6 rectangle is too big to be a hexomino envelope, as shown here.



The 3 × 3 square, however, is a hexomino envelope.



Name(s) ____________________

3. Here is a list of rectangles.

2 × 2	1 × 5	2 × 3	2 × 5
3 × 3	3 × 5	2 × 4	1 × 6
3 × 4	2 × 6	4 × 4	3 × 6

Experiment on grid paper, then perform the following three steps.

a. Cross out those rectangles in the list above that are too small to be hexomino envelopes.

b. Cross out those rectangles in the list above that are too big to be hexomino envelopes.

c. Make sure the remaining rectangles in the list are hexomino envelopes!

4. Find as many hexominoes as you can. Sketch them on grid paper.

Discussion

A. In some cases, the perimeter of the envelope is the same as the perimeter of the corresponding polyomino. Is it ever true that the perimeter of the envelope is greater than that of the polyomino? Is it ever true that it is less? Explain.

B. Which hexominoes can be folded into a cube?

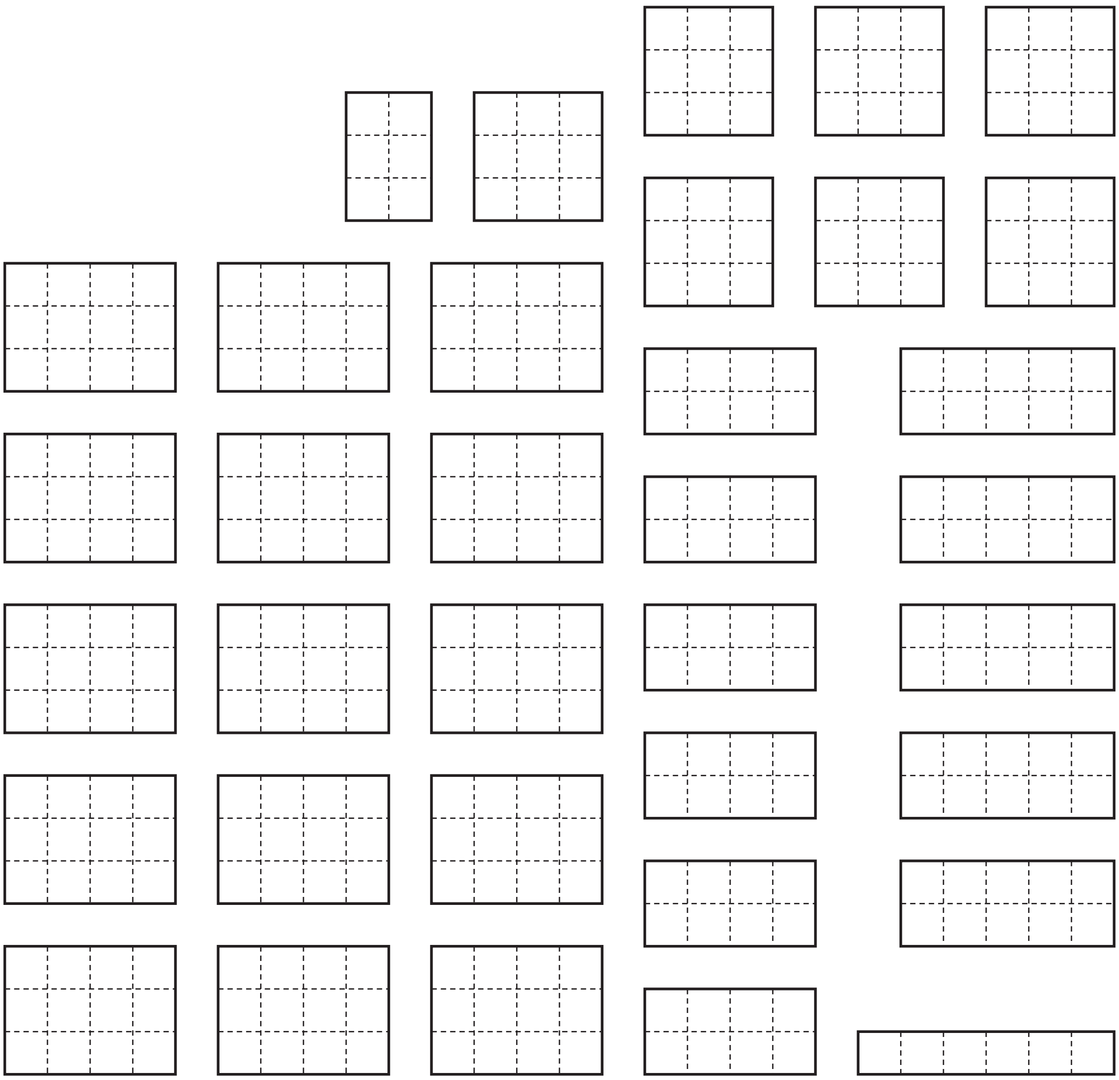
LAB 4.6

Name(s) ____________________

Classifying the Hexominoes

- **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

There are 35 hexominoes. Is that the number you found in Lab 4.5? Organize them by drawing them in the envelopes below. As you work, watch out for duplicates. There should be no empty envelopes when you finish.

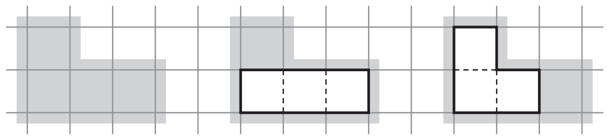


LAB 4.7
Minimum Covers

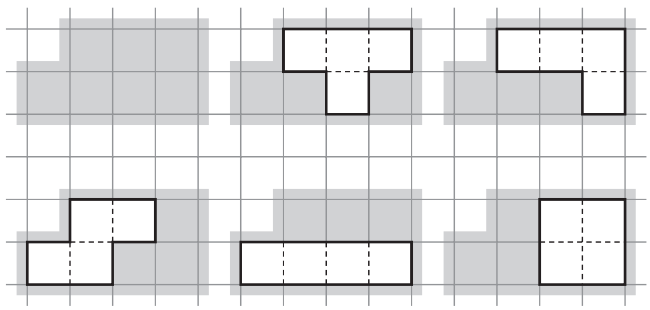
Name(s) ______________________

■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

The squares covered by shading below form the smallest shape onto which you can fit either tromino. Its area is 4 square units. This shape is called a *minimum tromino cover.*



All of the tetrominoes fit on the shape covered by shading below, which has an area of 7 square units. It is not, however, the minimum tetromino cover.



1. Find the smallest shape onto which any of the tetrominoes will fit (the minimum tetromino cover).

a. Sketch it on grid paper.

b. What is its area?

2. Find the minimum pentomino cover.

a. Sketch it on grid paper.

b. What is its area?

3. Find the minimum hexomino cover.

a. Sketch it on grid paper.

b. What is its area?

Discussion

A. How did you find the minimum cover? How do you know that no smaller cover is possible?

B. Is the minimum cover unique? In other words, can more than one shape be used for a minimum cover?

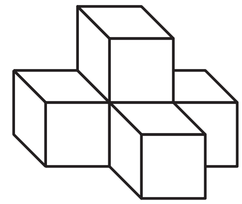
LAB 4.8

Name(s) ____________________________

Polycubes

■ **Equipment:** Interlocking cubes

Polycubes are three-dimensional versions of polyominoes. They are made by joining cubes face to face. At right is an example (a pentacube).



1. Use your cubes to make one monocube, one dicube, two tricubes, and eight tetracubes. **Hint:** Most of them look just like the corresponding polyominoes, except for three of the tetracubes, which are genuinely 3-D. Two of them are mirror images of each other.
2. Put aside all the polycubes that are box-shaped. The remaining pieces should have a total volume of 27. They can be combined to make the classic Soma® Cube.
3. Find other interesting figures that can be made using the Soma pieces.
4. Find all the pentacubes. Twelve look like the pentominoes, and the rest are genuinely 3-D.
5. What rectangular boxes can you make using different pentacubes?

Discussion

A. How did you organize your search for polycubes? (Did you use the polyominoes as a starting point? If yes, how? If not, what was your system?) How did you avoid duplicates?

B. What are some ways to classify the polycubes?

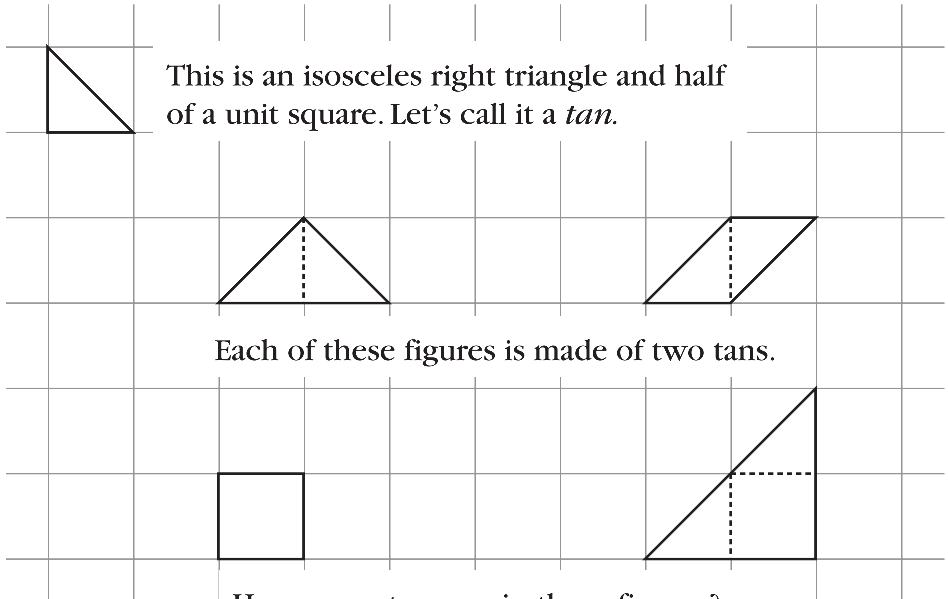
C. What would be the three-dimensional equivalent of a polyomino envelope?

D. Make a family tree for a tetracube.

LAB 4.9
Polytans

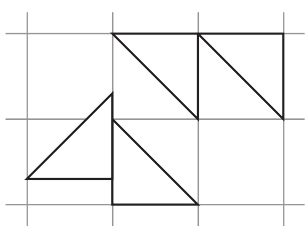
Name(s) ______________________

■ **Equipment:** Graph paper



How many tans are in these figures?
Draw diagonals to find out.
__________ and __________

1. There are only four figures that are made of three tans. Use graph paper to find them.



Note: Arrangements like these do not count. Tans must touch by whole sides.

2. Figures that are made of four tans are called superTangrams. How many of them can you find? Draw them on graph paper.

Discussion

A. What is your method for finding superTangram shapes?

B. How do you know when you have found all the superTangrams?

C. Find the perimeters of all the superTangrams and rank them from shortest to longest.

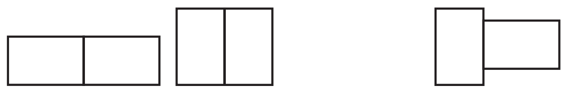
LAB 4.10

Name(s) ____________________________

Polyrectangles

■ **Equipment:** Template, graph paper

Polyrectangles are like polyominoes, except that their basic unit is a rectangle that is not a square. The rectangle edges that meet to form a polyrectangle must be equal. There is only one domino, but there are two directangles.

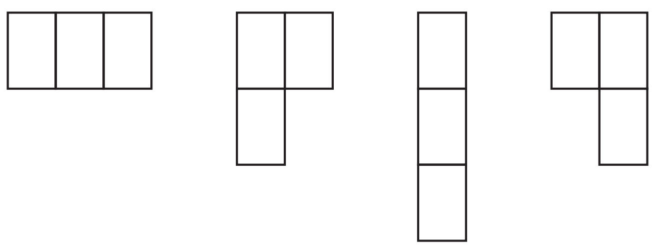


The two directangles **Not a directangle**

Record your findings for the following questions on a separate sheet of paper.

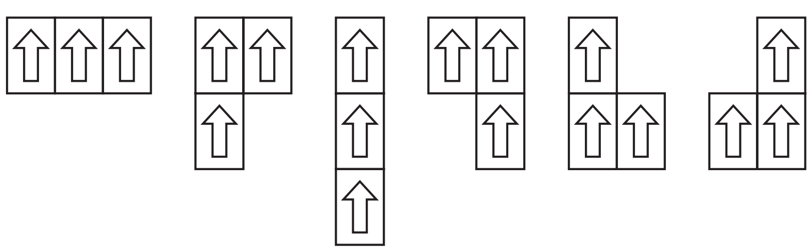
1. Find all the trirectangles.
2. Find all the tetrarectangles.
3. Find all the pentarectangles.

If we add the constraint that polyrectangles cannot be turned over, additional figures are needed. For example, there are four one-sided trirectangles, shown below. The second and fourth examples would be duplicate trirectangles, but as one-sided trirectangles they are different.



4. Find all the one-sided tetrarectangles.

Yet another variation is to think of polyrectangles as polystamps, torn out of a sheet of stamps, with each rectangle displaying a figure and some text. You cannot turn polystamps at all. Up is up. In that situation, we have six tristamps:



5. Find all the tetrastamps.

Discussion

A. Each polyrectangle corresponds to one polyomino. However, a polyomino may correspond to one or two polyrectangles. What determines whether it is one or two?

B. What is the relationship between polystamps and polyrectangles?

C. The more symmetries a polyomino has, the more polyrectangles, one-sided polyrectangles, and polystamps there are that are related to it. True or false? Explain.