CHAPTER 13

The spiraling vortex of a whirlpool

Coming in this chapter:

Exploration You want to make pens for Stripe, your pet zebra, and Polka Dot, your pet leopard. You have 100 feet of fencing. If you use all of it to make two pens of equal area, what is the biggest area possible?

MAKING DECISIONS

LESSON 13.1

Rectangular Pens: Constant Perimeter

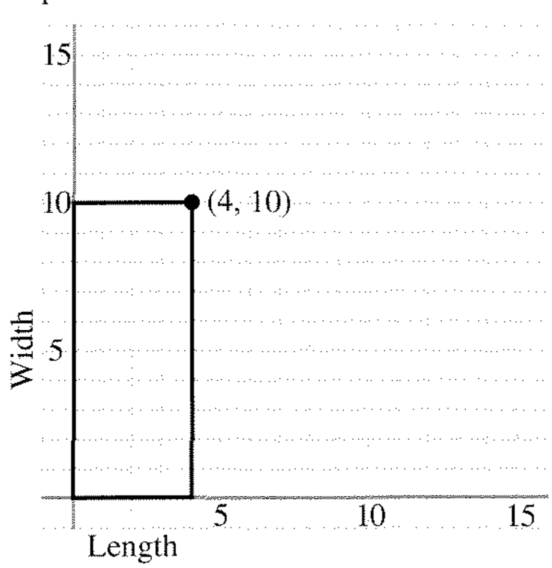
You will need:

- graph paper
- a graphing calculator (optional)

1. **Exploration** You want to make a rectangular pen for Stripe, your pet zebra. Even though Stripe takes many walks around town, you want to make sure she has as much space as possible inside the pen. You have 50 feet of fencing available. If you use all of it to make the pen, what is the biggest area possible? Find out by trying various dimensions for the pen.

WIDTH AS A FUNCTION OF LENGTH

You have 28 feet of fencing to make a rectangular pen. There are many possible dimensions for this pen. One possible pen, 10 feet wide by 4 feet long, is shown below. In this section you will investigate how the length and width change in relation to one another if you keep the perimeter constant.



2. a. On graph paper, draw axes and at least six pens having a perimeter of 28.
   b. The upper right corner of the pen in the figure has been marked with a • and labeled with its coordinates. Do this for the pens you drew. Then connect all the points marked with a •. Describe the resulting graph.
3. a. Make a table showing all the coordinates on your graph. Look for a pattern and make three more entries in the table.
   b. Write an equation for the function described by your graph and table.

Summary

4. The point whose coordinates are (4, 10) is on the graph.
   a. What does the sum of these numbers represent in this problem?
   b. What does the product represent?
5. a. What is the greatest possible length of a pen? How can you see this on your graph?
   b. How many rectangles are possible if the dimensions are whole numbers? How many are possible otherwise?
   c. Explain why the graph should not be extended into quadrants II and IV.
6. If you increase the length by one foot, does the width increase or decrease? Does it change by the same amount each time? Explain.

AREA AS A FUNCTION OF LENGTH

In the previous section you may have noticed that the area of the rectangles changed even though the perimeter remained constant. In this section you will investigate how the area changes as a function of length, if you keep the perimeter constant.

7. Write the area of the corresponding rectangle next to each of the points marked with a • on the graph from problem 2.

8. Make a graph of area as a function of length. Show length on the x-axis and area on the y-axis. Connect the points on your graph with a smooth curve. What kind of curve is it?

9.

a. Label the highest point on your graph with its coordinates. Interpret these two numbers in terms of this problem.

b. Where does the graph cross the x-axis? What do these numbers mean?

c. If you increase the length by one foot, does the area increase or decrease? Does it change by the same amount each time? Explain.

10. Summary

a. Describe in words how you would find the area of the rectangular pen having perimeter 28 if you knew its length.

b. If the perimeter of a rectangular pen is 28 and its length is L, write an algebraic expression for its area in terms of L.

c. If you had 28 feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

Generalizations

11. Say the perimeter of a rectangle is P and its length is L. Write the following expressions in terms of P and L. (A sketch may help.)

a. an expression for the width

b. an expression for the area

12. Explain how to find the length that gives the maximum area. Write an algebraic expression for it in terms of P only.

PARABOLAS THROUGH THE ORIGIN

13. Graph each of the following functions, using graph paper. Since you will want to compare your graphs in the end, use the same pair of axes for all your graphs. Use a scale that will show values from -5 to 20 for x and from -20 to 100 for y. This will allow you to see all four graphs clearly.

a. $y = x(8 - x)$ b. $y = x(15 - x)$

c. $y = x(12 - x)$ d. $y = x(20 - x)$

14. For each of the four parabolas in problem 13,

a. label the graph with its equation;

b. label the x-intercepts;

c. label the vertex.

15. Generalization

a. Describe the graph of a parabola having equation $y = x(b - x)$. Write expressions for the coordinates of its intercepts and vertex in terms of b.

b. Do these expressions work for negative values of b? Explain, using examples.

16. Graph.

a. $y = x(x - 8)$ b. $y = x(x - 15)$

c. $y = x(x - 12)$ d. $y = x(x - 20)$

17. How do the graphs differ from the ones in problem 3? Discuss the vertex and the intercepts.

18. Generalization

a. Describe the graph of a parabola having equation $y = x(x - q)$. Write expressions for the coordinates of its intercepts and vertex in terms of q.

b. Do these expressions work for negative values of q? Explain, using examples.

19. Graph $y = ax(x - 3)$ for:

a. $a = 1$ b. $a = -1$

c. $a = 2$ d. $a = -3$

20. What is the effect of a on the position of:

a. the vertex?

b. the x-intercepts?

Find equations of the form $y = ax(x - q)$ for parabolas *through the origin*, with the given x-intercept and the vertex with the given y-coordinate.

		x-intercept	**y-coordinate of vertex**
21.	a.	4	4
	b.	4	8
	c.	4	2
	d.	4	-6
22.	a.	8	4
	b.	2	4
	c.	-4	4
	d.	-6	-6

REVIEW FIXED POINTS

23. Find the fixed point for the function $y = 6x + 8$.

24. Solve the system: $\begin{cases} y = 6x + 8 \\ y = x \end{cases}$

25. Explain the statement: *To find the fixed points of a function, find the intersection of its graph with the line y = x.*

26. Test whether the statement is true by finding the fixed points of $y = x^2$.

LESSON 13.2

Advanced Penmanship

You will need:

graph paper

PEN PARTITIONS

Assume that you have 50 feet of fencing to build a rectangular pen. You plan to use the garage wall as one side of the pen, which means you need to use your fencing for only three of the four sides. Since you are considering adopting more pets, you want to investigate what happens when you use some of the fencing to divide the pen into two or more parts by building partitions inside the pen, at a right angle to the wall.

1. Make a rough sketch of what this pen might look like,
   a. having no internal partitions;
   b. divided into two sections.

2. With no partitions, is it possible to get a square pen? If so, what are its dimensions?

3. With one partition, is it possible to get two square sections? If so, what are their dimensions?

Call the side of the pen parallel to the wall the *length,* and the distance between the wall and the side opposite the wall x.

4. Imagine you are dividing the pen into two parts. Make a table having three columns: x, the length, and the total area of the pen.

Generalizations

5. Look for patterns in your table. Express algebraically as functions of x,
   a. the length; b. the area.

6. What is the equation that expresses the length as a function of x, if the pen is divided into the given number of parts. (Make sketches. If you need to, make tables like those in problem 4.)
   a. 1 b. 3
   c. 4 d. n

7. Repeat problem 6, but this time find the area as a function of x.

GRAPHS OF AREA FUNCTIONS

This section is about the graphs of functions like the ones you found in problem 7.

8. Using graph paper, graph each of the following functions. To make comparison easier, use the same graph, or at least the same scale, for all your graphs. To see all four graphs clearly, use a scale that will show values from $^-5$ to 15 for x and from $^-50$ to 50 for y. When making a table of values, use both negative and positive values for x. Keep these graphs, because you will need them in the next section.
   a. $y = x(12 - x)$
   b. $y = x(12 - 2x)$
   c. $y = x(12 - 3x)$
   d. $y = x(12 - 4x)$

9. For each graph,
   a. label the graph with its equation;
   b. label the x-intercepts;
   c. label the vertex.

10. Write a brief description comparing the four graphs. Describe how the graphs are the same and how they are different.

11. Without graphing, guess the vertex on the graph of $y = x(12 - 6x)$. Explain how you arrived at your guess.

DIFFERENT FORMS

As you learned in Lesson 1 the equations of parabolas through the origin can be written in the form $y = ax(x - q)$.

12. For each parabola described in (a-d), find a function of the form $y = ax(x - q)$:

a. x-intercepts: 0 and 12, vertex: (6, 36)
b. x-intercepts: 0 and 6, vertex: (3, 18)
c. x-intercepts: 0 and 4, vertex: (2, 12)
d. x-intercepts: 0 and 3, vertex: (1.5, 9)

13. How are the intercepts and the vertex related to the values of a and q in the equation $y = ax(x - q)$?

The equations in problems 8 and 12 have the same graphs. You can verify this by checking that they have the same vertices and intercepts, and in fact that for any x they yield the same y. In other words, the equations are equivalent. We can use the distributive law to confirm this. For example, for problem 8a:

$$y = x(12 - x) = 12x - x^2$$

And for problem 12a:

$$y = -x(x - 12) = -x^2 + 12x$$

14. Show that the other three pairs of equations in problems 8b-d and 12b-d are equivalent.

It is possible to convert equations like the ones in problem 8 to the form $y = ax(x - q)$ by factoring. For example:

$$x(24 - 6x) = 6x(4 - x) = -6x(x - 4)$$

15. Fill in the blanks:

a. $x(24 - 2x) = 2x(\_\_\_\_\_\_)$
b. $x(24 - 3x) = -3x(\_\_\_\_\_\_)$
c. $x(24 - 4x) = \_\_\_\_(x - 6)$

16. Write in the form $y = ax(x - q)$ and find the vertex and the intercepts.

a. $y = x(12 - 6x)$
b. $y = x(50 - 5x)$
c. $y = x(50 - 3x)$
d. $y = x(50 - (n + 1)x)$

MAXIMIZING AREA

17. If you have to use part of the 50 feet of fencing for a partition to divide the pen into two equal parts, what is the largest total area you can get for the enclosure? Explain how you got your answer, including a sketch and graph if necessary.

18. Solve problem 17 if you want to divide the pen into three equal parts.

19. Solve problem 17 if you want to divide the pen into n equal parts.

20. Look at your solutions for problems 17, 18, and 19. In each case look at the shapes of the subdivisions of the pen having the largest area. Are they always squares? Are they ever squares? Does the answer to this depend on the value of n? Explain.

21. Look at your solutions for problems 17, 18, and 19. In each case look at how much of the fencing was used to construct the side parallel to the garage for the pen having maximum area. What fraction of the fencing was used to construct this side? Does the answer depend on the value of n? Explain.

22. **Report** Imagine you are the representative of a fencing company presenting information to a customer. Write a complete illustrated report, making clear who the customer is and what the pens are needed for. Explain how to maximize the area of the pens for a given amount of fencing. Discuss both divided and undivided pens.

LESSON 13.3

The Zero Product Property

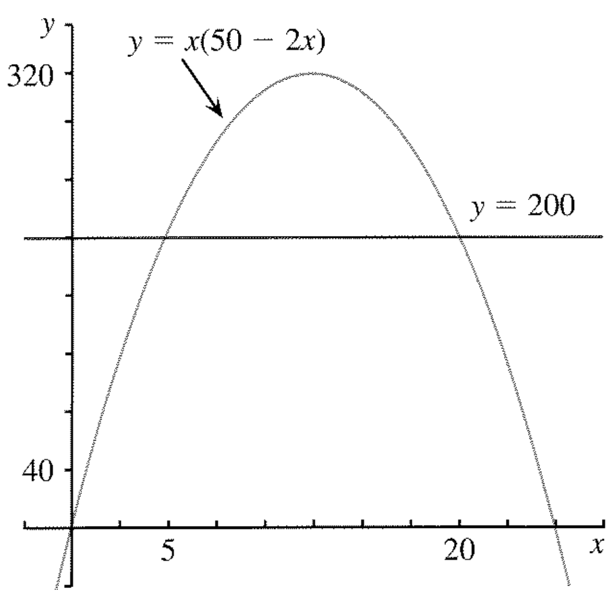
You will need:

graph paper

or a graphing calculator (optional)

1. Given that you have 50 feet of fencing and that you can use the wall of the garage for the fourth side of your pen, what dimensions should you choose to make a rectangular pen having area 200 square feet? Solve by trial and error or by graphing. (There is more than one solution.)

This problem can be solved by writing the equation $x(50 - 2x) = 200$, where x is the distance from the wall to the side opposite it. One way of doing it is to find the intersection of the graphs of $y = x(50 - 2x)$ and $y = 200$.



2. Use this method to find the dimensions for the following areas:

a. 300 b. 250

Another method of solving this kind of problem is to write a quadratic equation and factor it, as explained in the following sections.

ZERO PRODUCTS

3. If $ab = 0$, which of the following is impossible? Explain.

a. $a \neq 0$ and $b \neq 0$

b. $a \neq 0$ and $b = 0$

c. $a = 0$ and $b \neq 0$

d. $a = 0$ and $b = 0$

Zero Product Property: When the product of two quantities is zero, one or the other quantity must be zero.

An equation like $(x + 6)(2x - 1) = 0$ can be solved using the zero product property. Since the product in the equation is zero, you can write these two equations.

$$x + 6 = 0 \quad \text{or} \quad 2x - 1 = 0$$

4. You know how to solve these equations. Write the solutions.

5. There are two solutions to the equation $(x + 6)(2x - 1) = 0$. What are they?

Solve these equations.

6. $(3x + 1)x = 0$

7. $(2x + 3)(5 - x) = 0$

8. $(2x - 2)(3x - 1) = 0$

SOLVING QUADRATIC EQUATIONS

Some quadratic equations can be solved using the zero product property.

Example: Find the values of x for which

$$x^2 + 6x = -5.$$

First rewrite the equation so you can apply the zero product property.

$$x^2 + 6x + 5 = 0$$

In factored form, this is written:

$$(x + 5)(x + 1) = 0.$$

Since the product is 0, at least one of the factors must be 0. So $x + 5 = 0$ or $x + 1 = 0$.

9. What are the two solutions of the equation $(x + 5)(x + 1) = 0$?

Example:

Find the values of x for which $6x^2 = 12x$.

First rewrite the equation so that you can apply the zero product property.

$$6x^2 - 12x = 0$$

In factored form, this is written:

$$6x(x - 2) = 0.$$

10. What are the two solutions to the equation $6x(x - 2) = 0$?

11. Factor and use the zero product property to solve these quadratic equations.

a. $x^2 - x = 2$

b. $2L^2 - L = 3$

c. $W^2 + 10W + 16 = 0$

d. $3M^2 + 30M + 48 = 0$

To solve problem 2a, write the equation:

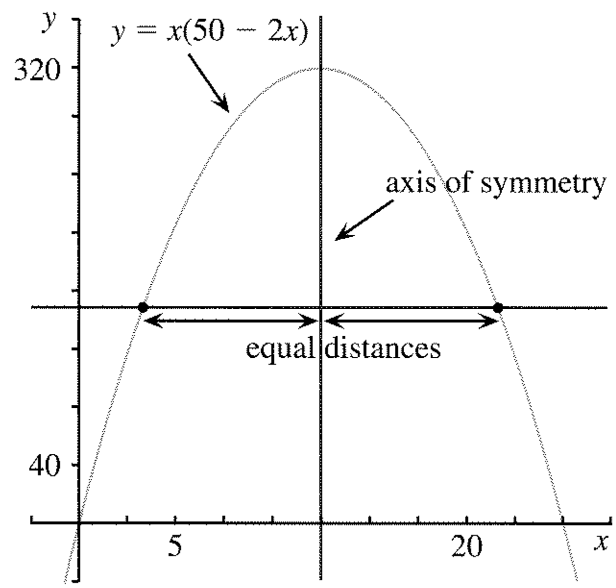
$$x(50 - 2x) = 300$$
$$-2x^2 + 50x - 300 = 0$$
$$2x^2 - 50x + 300 = 0$$
$$x^2 - 25x + 150 = 0$$

12. Explain the four steps.

13. Factor the final equation and use the zero product property to solve it.

Unfortunately, most quadratic equations cannot easily be solved by factoring. In the next chapter you will learn a way that always works to solve quadratic equations.

SYMMETRY



The vertical line through the vertex of a parabola is called its *axis of symmetry.*

14. How far is each x-intercept from the axis of symmetry in the preceding graph?

The x-intercepts are *equidistant* from the axis of symmetry. (They are at an equal distance from it.) As you can see in the figure, this is also true of any pair of points of the parabola that lie on the same horizontal line as each other.

15. Refer to the graph for problem 1.

a. Show that the two solutions to problem 1 are equidistant from the axis of symmetry.

b. Is this also true of the two solutions to problem 2a? What about problem 2b? Show your work.

VERTEX AND INTERCEPTS

In an equation like $y = 2(x + 3)(x - 4)$, you can quickly find the intercepts and the vertex.

16. What is the value of x at the y-intercept? Substitute this value for x in the equation and find the y-intercept.
17. What is the value of y at the x-intercepts? Substitute this value for y in the equation and find the x-intercepts with the help of the zero product property.
18. If you know the x-intercepts, how can you find the x-coordinate of the vertex? Find it.
19. If you know the x-coordinate of the vertex, how can you find its y-coordinate? Find it.
20. Find the intercepts and vertex for:
    a. $y = 0.5(x - 0.4)(x - 1)$
    b. $y = 2(x + 3)(x + 4)$
21. Explain how you would find the intercepts and vertex for a function of the form
$$y = a(x - p)(x - q).$$
22. Find the equation and the vertex for a parabola having the following intercepts:
    a. (3, 0), (6, 0), (0, 36)
    b. (3, 0), (6, 0), (0, 9)
    c. (-3, 0), (-6, 0), (0, -9)
    d. (-3, 0), (6, 0), (0, 6)
23. The vertex and one of the two x-intercepts of parabolas are given. Find the equation and the y-intercept.
    a. vertex: (2, -2); x-intercept: (1, 0)
    b. vertex: (1, -12); x-intercept: (-1, 0)
    c. vertex: (3, 4.5); x-intercept: (6, 0)

DISCOVERY TWO DEFINITIONS

Definition: The absolute value of a number is the distance between that number and zero.

Browsing through Ginger's calculus book, Mary and Martin noticed this definition:

Definition: $|x| = \begin{cases} x \text{ for } x > 0 \\ -x \text{ for } x < 0 \end{cases}$

"That $-x$ must be a misprint," Mary commented. "Absolute value can't be negative."

24. **Report** Write a letter to Mary explaining everything you know about absolute value. Restate the two definitions presented above in your own words. Using examples, explain why they are equivalent, and why Mary was wrong about the misprint.

LESSON 13.4

Rectangular Pens: Constant Area

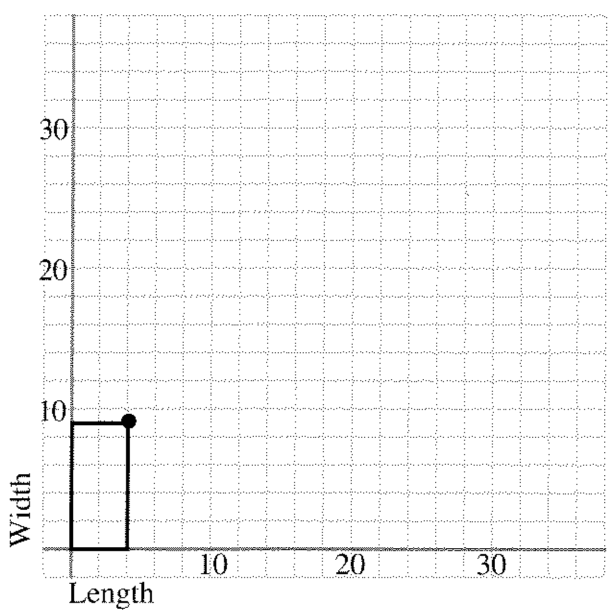
You will need:

- graph paper
- a graphing calculator (optional)

1. **Exploration** You bought 45 square feet of artificial turf for the floor of Stripe's backyard. You can cut it up any way you like, but you want to use all of it. Since you're almost broke (artificial turf is expensive) you would like to spend as little money as possible on fencing. What's the least amount of fencing you could buy and still make a rectangular pen that surrounded the artificial turf on all four sides? Find out by trying various dimensions for the pen.

WIDTH AS A FUNCTION OF LENGTH

Suppose you want to make a rectangular pen having area 36.



2. a. On graph paper draw a pair of axes and show five rectangular pens that would have an area of 36 square feet. The lower left corner should be at the origin. An example is shown in the figure.

b. Mark with a • the upper right corner of each rectangle you drew. Then write in the coordinates of each of these points.

c. Connect the •s. Do they lie in a straight line or on a curve? Describe any patterns you notice.

3. Make a table showing some of the coordinates on your graph. Look for a pattern in your coordinates and make three more entries in the table.

4. Write an algebraic equation that expresses the width as a function of the length.

5.

a. Would it be possible to have a pen having length greater than 30? 32? 36? Explain your answers, giving examples.

b. Explain why your graph will never touch the x-axis or the y-axis.

c. If you increase the length by one foot, does the width increase or decrease? Does it change by the same amount each time? Explain.

PERIMETER LINES

In the previous section you probably noticed that the perimeter of the rectangles changed even though the area remained constant. In this section you will investigate how the perimeter varies as a function of length if you keep the area constant.

6. Write the perimeter of the corresponding rectangle next to each • you marked on the graph. Look for patterns.

Your graph should show pairs of points that correspond to the same perimeter. For example, (3, 12) and (12, 3) both correspond to the perimeter 30.

7. Connect (3, 12) and (12, 3) to each other by a straight line. Extend it to its intercepts. Interpret the intercepts in terms of this problem.

8. On your graph find two points that both correspond to a perimeter of 26. Repeat problem 7 for these points. Then find other pairs of points that both correspond to the same perimeter and repeat problem 7 for each of these pairs. What patterns do you see?

9. Use the graph to estimate the dimensions of a rectangle having area 36 and perimeter 36.

PERIMETER AS A FUNCTION OF LENGTH

10. Make a graph of perimeter as a function of length. Show length on the x-axis and perimeter on the y-axis. Connect the points on your graph with a smooth curve. Describe the shape of the curve.

11. Label the lowest point on your graph with its coordinates. Interpret these two numbers in terms of the problem.

Note: The graph is *not* a parabola, and its lowest point is *not* called a *vertex*.

12. Explain why your graph will never touch the x-axis or y-axis.

13. If you increase the length by one foot, what happens to the perimeter? Can you tell whether it will increase or decrease? Does it increase or decrease by the same amount each time? Explain.

14. **Summary**
    a. For a fixed area of 36 square feet, explain in words how you would find the perimeter of the rectangular pen if you were given the length.
    b. If the area of a rectangular pen is 36 and its length is L, write an algebraic expression for its perimeter.
    c. If you had to enclose a rectangular area of 36 square feet and wanted to use the least amount of fencing, what would the length, width, and perimeter be? Explain.

Generalizations

15. If the area of a rectangular pen is A and its length is L,
    a. write an algebraic expression for its width in terms of A and L;
    b. write an algebraic expression for its perimeter in terms of A and L.

16. Explain how to find the length that gives the minimum perimeter. Write an algebraic expression for it in terms of A only.

NUMBER PUZZLES

17. Find two numbers x and y whose product is 75 and whose sum is 20. Explain your method.

18. Graph the equations $xy = 75$ and $x + y = 20$ on the same pair of axes. Find their point of intersection. How is this point related to your answer to problem 17?

19. Find two numbers whose product is 75 and whose sum is 23.75.

20. If two numbers have a product of 75, what is the smallest value their sum could take? What is the largest? Explain.

THINKING WRITING 13.A Business Applications

MAXIMUM PROFIT

The Widget Company was trying to sell a widget for $24, but no one was buying. They decided to try to attract customers by reducing their prices. They found that for every $1 they lowered the price, they attracted ten customers.

Price Reduction	Price	# of Customers	Gross Profit
$0	$24	0	$0
$1	$23	10	$230
$2	$22	20	$440

1. a. Copy and extend the table for at least eight possible price reductions.
   b. If the price is $14, how many people will buy a widget? What will the gross profit be?
   c. If the price is lowered by $\$x$, how many people will buy a widget? What will the gross profit be?
   d. Make a graph showing how the gross profit depends on the price reduction. Put the price reduction on the x-axis and the profit on the y-axis.
   e. Interpret your graph. What price gives the most profit? Explain.
   f. Write an equation for your graph.

The Widget Company was trying to sell an item for P dollars, and no one was buying it. They found that for every $1 they lower the price, they gain C customers.

2. If they lower the cost by x and the gross profit is y, write an equation for y in terms of x.

3. Write an algebraic expression for:
   a. the amount by which the price should be reduced in order to maximize the profit;
   b. the maximum profit possible.

MINIMUM COST

The Widget Company would like to ship 2000 widgets. They must be packaged in boxes of equal weight. (Each widget weighs one pound.) The L.A. Barge Company charges a basic rate of $100 per box for shipping. It also adds a surcharge to the total cost of the shipment that depends on the weight of the individual boxes, at the rate of $1 per pound.

Example: If the widgets are packed in 10 boxes, each will weigh 200 lbs.

Basic charge	Surcharge	Total
10 boxes · $100 per box	$200	$1200

4. Explain, using examples of possible ways to package the 2000 widgets, how the L.A. Barge Company's policy guarantees that customers will not ship their goods in too many boxes, or in boxes that are too heavy.

5. Write an algebraic expression for the cost of shipping the 2000 widgets, in terms of the number of boxes.

6. What is the number of boxes that would be the cheapest way to ship the widgets? Explain how you get your answer. (Hint: You may use trial and error or graphing.)

7. Using the cheapest way, how much does it cost per widget?

8. Report Imagine you work for the Widget Company. Prepare an illustrated report to other employees about:
   a. the pricing of widgets and how to maximize profits, and
   b. the shipping of widgets and how to minimize cost.

LESSON 13.5

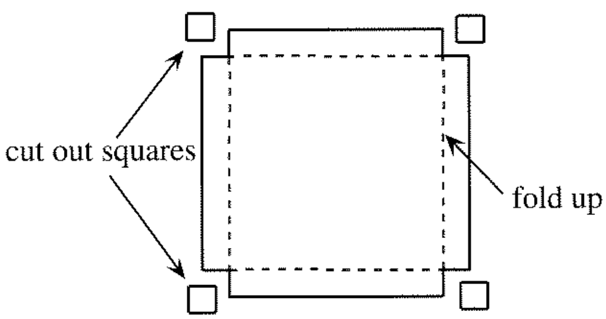
Packing and Mailing

You will need:

- graph paper
- centimeter grid paper
- scissors
- tape

MAXIMIZING VOLUME

You can make cardboard trays to hold 1-cm^3 cubes. Start with an 18-cm-by-18-cm piece of grid paper. Cut a square out of each corner and fold up the sides to form a tray.



1. **Exploration** Work with other students to make as many different trays as you can by cutting square corners out of an 18-cm-by-18-cm piece of paper or cardboard. Figure out which tray holds the most cubes.

2. Make a table showing the side of the square corner that was cut out, the area of the base, and the number of cubes the tray would hold. (For example, if a 2-by-2 square is cut out at each corner, the area of the base should be 196 cm^2, and the tray should hold 392 cubes.)

3. If the side of the square cut out of the corner is x,
   a. what is the area of the base?
   b. what is the volume of the tray?

4. Make a graph of the volume of the tray as a function of x. Include some fractional values of x.

5. What is the height of the tray that will give the maximum volume?

6. What are the x-intercepts of the graph? Interpret them in terms of this problem.

7. Draw a vertical line through the highest point on the graph. Are the x-intercepts equidistant from it?

8. Extend the graph in both directions by using a few more values for x beyond the x-intercepts.

9. Explain why the points you added in problem 8 do not represent the tray problem.

10. Is the graph a parabola? Explain, giving as many reasons as you can for your answer.

11. **Generalization** Find the height which would give the maximum volume if the initial piece of paper had the following dimensions. You may want to use tables of values.
    a. 12 by 12 b. S by S

STORING CUBES

Suppose you want to make a cardboard tray for storing 100 centimeter cubes. The base does not have to be square.

12. Exploration What should the dimensions of the tray be so it will contain the least cardboard? Explain.

13. Repeat problem 12 for:

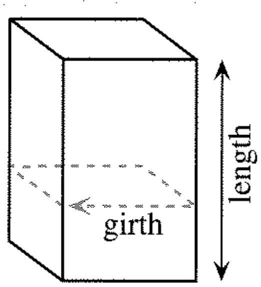
a. 50 cubes; b. 200 cubes;

c. 500 cubes; d. 1000 cubes.

14. Project Write an illustrated report explaining a strategy for solving this problem for N cubes.

POSTAL REGULATIONS

The U.S. Postal Service will not mail by Priority Mail™ anything that weighs more than 70 pounds or exceeds 108 inches in combined length and girth. (The girth is the distance around, as shown in the figure.)



15. Exploration Find the dimensions for a box that would satisfy the Priority Mail™ requirements and would hold as large a volume as possible.

16. Suppose you want to mail a box full of 20-inch-long dowels. What are the dimensions of the rectangular box having the largest volume that would satisfy postal regulations and would accommodate the dowels?

17. Repeat problem 16, this time for 12-inch-long dowels.

18. Project A lumber company needs to pack dowels in boxes that can be sent by Priority Mail™. Boxes need to be designed to ship dowels of each length. Explain, with examples, how to find dimensions for such boxes that will allow the packing of the maximum number of dowels.

19. A shipping company has the following rules:

- maximum length: 108 inches
- maximum length plus girth: 130 inches

In addition, they recommend two inches cushioning on all sides for fragile items. What is the largest volume possible for the contents of the package in the case of fragile items?

LESSON 13.6

Solving with Squares

You will need:

the Lab Gear

In this chapter you have used quadratic functions to solve problems involving finding a maximum area. In the next chapter you will be faced with problems for which it will be useful to solve quadratic equations. In this lesson we start to prepare for this.

EQUAL SQUARES

In Chapter 7, Lesson 7, you solved quadratic equations using the equal squares method. Some problems are easy to solve this way. For example,

$$x^2 - 10x + 25 = 16$$

can be written

$$(x - 5)^2 = 4^2$$

with a perfect square on each side.

1. Solve this equation. (Remember: There are two solutions.)

It is not necessary for the number on the right to be a perfect square, since you can take the square root of any nonnegative number.

Example:

$$x^2 - 10x + 25 = 7$$
$$(x - 5)^2 = 7$$
$$x - 5 = \sqrt{7} \text{ OR } x - 5 = -\sqrt{7}$$
$$x = 5 + \sqrt{7} \text{ OR } x = 5 - \sqrt{7}$$

Using your calculator, you can find decimal approximations for the two solutions:

$$x \approx 7.646 \text{ or } x \approx 2.354.$$

Solve these equations using the equal squares method. First give exact answers (using radicals if necessary); then find decimal approximations. Not all are possible.

2. $x^2 - 10x + 25 = 8$

3. $y^2 + 6x + 9 = 15$

4. $x^2 - 24x + 144 = 12$

5. $4r^2 - 4r + 1 = 6$

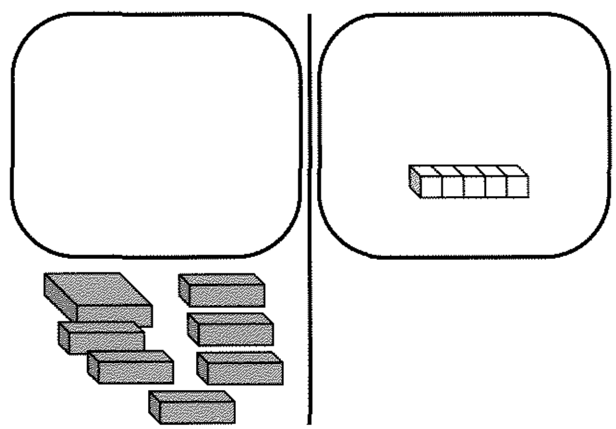
6. $9s^2 + 12s + 4 = 21$

7. $y^2 - 14y - 49 = -20$

COMPLETING THE SQUARE

In this section you will learn how to turn certain quadratic equations into equal squares equations that you know how to solve.

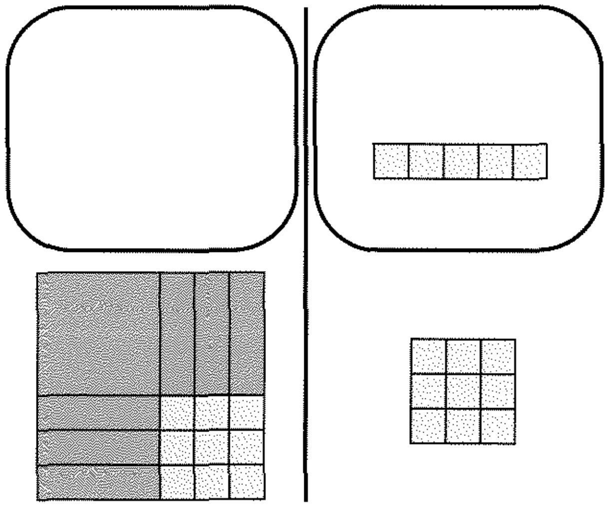
8. Write the equation shown by this figure.



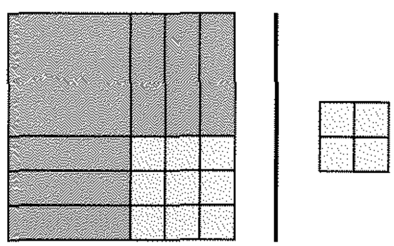
We will add the same quantity to both sides, so that the left side is a perfect square. This is called *completing the square*.

9. a. What number was added to both sides of the figure on the next page to make the left side a perfect square?

b. Write the resulting equation.



10. The right side can be simplified. The resulting equation is shown in the next figure. Write and solve this equation using the equal squares method. You should get two solutions.



Complete the square to solve these equations. You will need to rearrange blocks and add or subtract the same amount on both sides in order to get equal squares.

11. $x^2 + 2x - 3 = 0$ **12.** $x^2 + 12x = -11$

13. $x^2 + 4x = 0$ **14.** $x^2 + 10x - 6 = 5$

15. $x^2 + 8x = 20$ **16.** $x^2 + 6x + 9 = 25$

17. Generalization Explain how to figure out what number to add to both sides of an equation of the form $x^2 + bx = k$ in order to get an equal squares equation. Use sketches and examples.

SQUARE PRACTICE

Solve these equations by completing the square. Show all your work. Include a sketch showing the equal squares.

18. $x^2 + 8x = 33$ **19.** $x^2 + 4x = 96$

20. $x^2 + 6x = 55$ **21.** $x^2 + 10x = 56$

Solve these equations. Show your work.

22. $x^2 - 8x = 33$ **23.** $x^2 - 4x = 96$

24. $x^2 - 4x = -96$ **25.** $x^2 + x = 6$

Solve these equations. Show your work. Give exact answers, then find decimal approximations to the nearest hundredth.

26. $x^2 - 8x + 3 = 0$ **27.** $x^2 - 5x - 8 = 0$

28. $x^2 - 4x + 1 = 6$ **29.** $x^2 - 7x - 4 = 0$

QUADRATIC EQUATIONS CHECKPOINT

Solve two of these equations by factoring (and the zero product property), and two by completing the square.

30. $x^2 + 18x = 0$ **31.** $x^2 + 5x = 2.75$

32. $x^2 + 2x - 8 = 0$ **33.** $x^2 + 7x + 12 = 0$

While it is somewhat cumbersome, completing the square is an important technique when dealing with quadratic expressions. Unlike factoring, you can use it to solve any quadratic equation. In addition, we will use completing the square repeatedly to get more understanding of quadratic functions and to develop more efficient ways to solve quadratic equations.

34. Find a quadratic equation having solutions 5 and -2.

35. Find a quadratic equation having solutions $2 + \sqrt{5}$ and $2 - \sqrt{5}$.

36. Multiply. $(x - (4 + \sqrt{3}))(x - (4 - \sqrt{3}))$ (Hint: Carefully remove the inside parentheses and then set up a three-by-three multiplication table.)

37. You should have obtained a quadratic expression in problem 36. Set it equal to zero, and solve the equation.

LESSON 13.7

Finding the Vertex

You will need:
- graph paper
- graphing calculator (optional)

In this lesson you will learn how to find the vertex of graphs of quadratic functions. This will help you solve quadratic equations.

TRANSLATING A PARABOLA

Graph these functions on the same pair of axes. Use graph paper, even if you have a graphing calculator. For each one:

a. Graph the parabola.

b. Indicate the axis of symmetry with a dotted line, and label it with its equation.

c. Label the vertex with its coordinates.

1. $y = x^2 - 5$ **2.** $y = x^2 - 4x + 4$

3. $y = x^2 - 4x - 1$

Definition: The graphs obtained by shifting the location of a given graph without changing its shape are called *translations* of the original graph.

The graphs you drew in problems 1 through 3 are all translations of the graph of $y = x^2$.

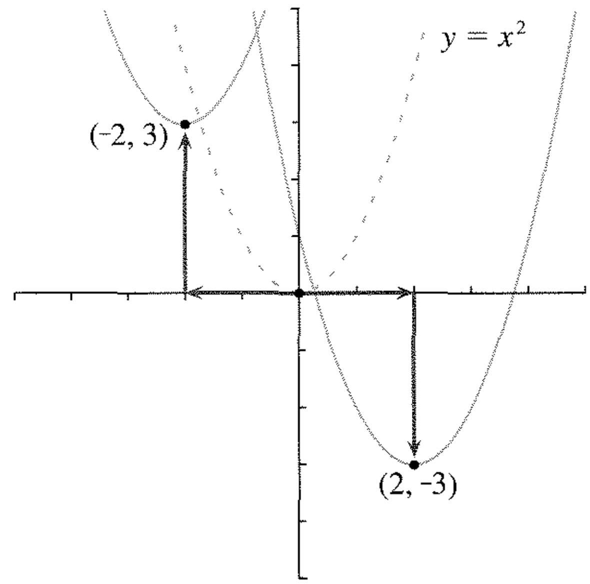
4. Which of the graphs you drew in problems 1 through 3 was obtained by shifting $y = x^2$

a. horizontally? b. vertically?

c. both horizontally and vertically?

VERTEX FORM

The vertex of the graph of $y = x^2$ is (0, 0). When the graph is shifted, the vertex is (H, V).



- If V is positive, the parabola $y = x^2$ has been shifted up; if V is negative, it has been shifted down.
- If H is positive, the parabola $y = x^2$ has been shifted to the right; if H is negative, it has been shifted to the left.

The graph of each function below is a translation of $y = x^2$. For each function:

a. Make a rough sketch of the graph.

b. Show the translation with arrows, as in the preceding figure.

c. Label the vertex with its coordinates.

(If you have a graphing calculator, use it for these problems. However, you should record the graphs with sketches on graph paper.)

5. $y = x^2 + 4$ **6.** $y = (x - 6)^2 - 4$

7. $y = (x + 6)^2 - 4$ **8.** $y = (x + 6)^2 + 4$

9. $y = (x - 6)^2$ **10.** $y = (x - 6)^2 + 4$

11. Write the equation of a parabola that is a translation of $y = x^2$ and has

a. a vertical distance of 8 and a horizontal distance of -3 (H = -3, and V = 8);

b. a vertical distance of -4 and a horizontal distance of 5;

c. 6 units to the left and 5 units down;

d. 3 units to the right.

Earlier in this chapter you looked at equations of parabolas having the form $y = a(x - p)(x - q)$. That form was convenient for finding x-intercepts.

12. Explain why the equations in problems 5-10 are in a form that makes it convenient to find the vertex by just looking at the equation.

The quadratic function $y = (x - H)^2 + V$ is said to be in *vertex form.*

13. Explain why the H in the vertex form equation is preceded by a minus, while the V is preceded by a plus.

14. The graph of $y = x^2$ meets the x-axis in one point. Give examples of translations of $y = x^2$ that meet the x-axis in the given number of points. Include explanations of how you chose different values of H and/or V.

a. 0 points   b. 1 point

c. 2 points

SITTING ON THE x-AXIS

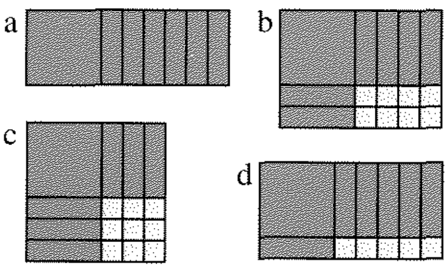
The quadratic function $y = x^2 + bx + c$ is said to be in *standard form.*

For problems 15-21, consider these five equations:

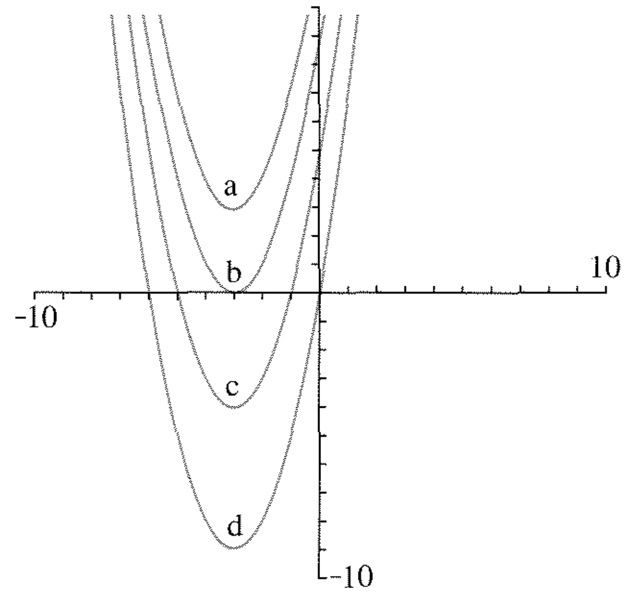
$y = x^2 + 6x$ $y = x^2 + 6x + 5$

$y = x^2 + 6x + 8$ $y = x^2 + 6x + 9$

$y = x^2 + 6x + 12$



15. Match each Lab Gear figure with an equation from the list of five.



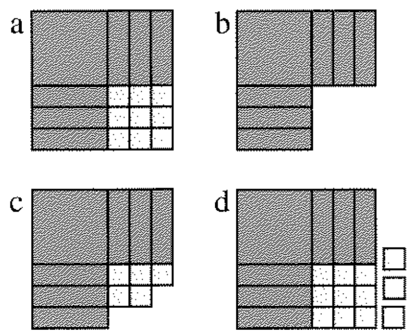
16. Match each parabola with an equation from the list of five.

17. Explain how to identify the parabolas with the help of:

a. the y-intercepts;

b. the Lab Gear figures, combined with the x-intercepts and the zero product property.

18. Explain why the graphs of perfect square quadratic equations have their vertices on the x-axis. (Hint: What is V?)



19. Match each Lab Gear figure with the corresponding equation from the five given earlier.

20. Find V for each equation of the five.

21. Explain how you can find V,

a. by looking at the Lab Gear figure;

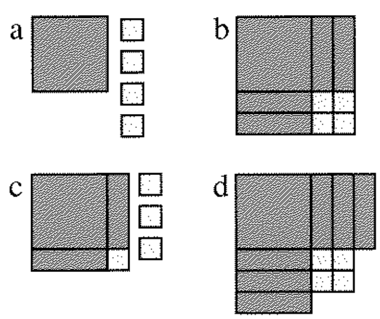
b. by looking at the equation.

STRADDLING THE y-AXIS

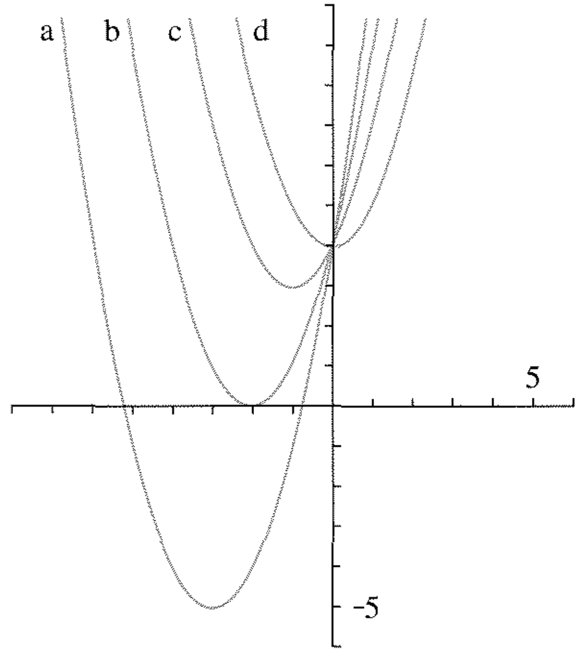
For problems 22-27, consider these four equations:

$y = x^2 + 4$ $\qquad$ $y = x^2 + 2x + 4$

$y = x^2 + 4x + 4$ $\qquad$ $y = x^2 + 6x + 4$



22. Match each Lab Gear figure with the corresponding equation from the four.



23. Match each equation with the correct graph.

24. Explain why the graphs of equations of the form $y = x^2 + c$ have their vertex on the y-axis. (Hint: What is H?)

25. Find H for each equation in the list of four.

26. Explain how you can find H,

a. by looking at the Lab Gear figure;

b. by looking at the equation.

27. a. What is H for any graph of an equation of the form $y = x^2 + 16x + c$?

b. What is H for any graph of an equation of the form $y = x^2 - 16x + c$?

28. **Generalization** Explain why, for graphs of equations in the form $y = x^2 + bx + c$, $H = -(b/2)$.

29. **Report** Write an illustrated report explaining how to find the vertex of a parabola if the equation is in:

a. the form $y = (x - p)(x - q)$;

b. vertex form; $\qquad$ c. standard form.

LESSON 13.8

Quadratic Equations: $x^2 + bx + c = 0$

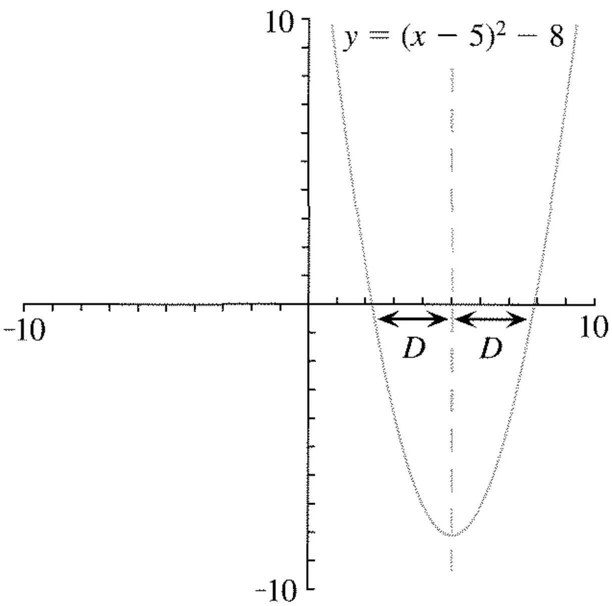
You will need:

graph paper

graphing calculator (optional)

FINDING THE x-INTERCEPTS

Earlier in this chapter you learned how to find the vertex after finding the x-intercepts. In this section you will learn how to find the x-intercepts after finding the vertex.



1. **Exploration** As the figure shows, the x-intercepts are equidistant from the axis of symmetry. How can you tell how far they are from it? That distance is indicated by D on the figure. Is it possible to know the value of D by looking at the equation? Try several values for H and V in equations having the form $y = (x - H)^2 + V$. Look for a pattern.

The graph of each of the following quadratic functions is a translation of $y = x^2$. For each function:

a. Make a rough sketch of the graph. Draw and label the axis of symmetry.

b. Find H and V.

c. Find exact values, not approximations, for the x-intercepts. (Set $y = 0$ and use the equal squares method.)

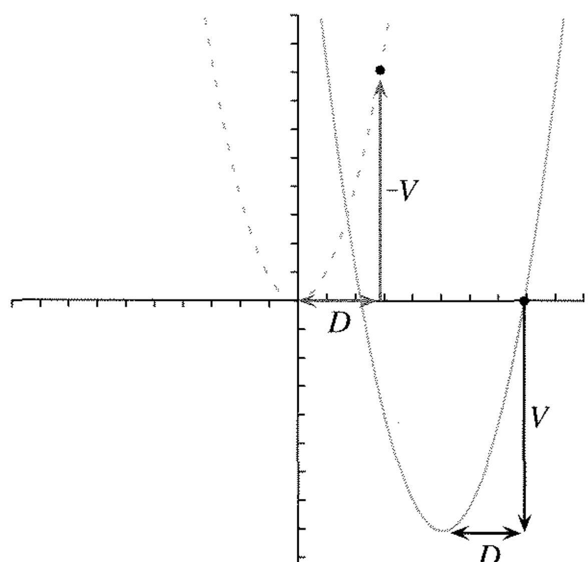
d. Find D, the distance of each x-intercept from the line of symmetry.

2. $y = x^2 - 9$ **3.** $y = (x - 5)^2 - 9$

4. $y = (x - 9)^2 - 5$ **5.** $y = (x + 9)^2 - 5$

6. $y = (x + 9)^2 + 5$ **7.** $y = (x - 9)^2$

8. Use patterns in problems 2-7 to explain how D and V are related.



This figure shows D and V on a parabola that was translated from $y = x^2$. In this example, V was a negative number, and the translation was in a downward direction. The arrows representing D and V are also shown on the original parabola. (On $y = x^2$, the direction of the arrow for V was reversed. What is shown is actually the opposite of V. This is indicated by the label $-V$. Since V is negative, $-V$ is positive.)

9. Use the figure to explain why $-V = D^2$, and therefore $D = \sqrt{-V}$.

10. Summary Explain why the x-intercepts, when they exist, are equal to $H - \sqrt{-V}$ and $H + \sqrt{-V}$.

SOLVING QUADRATIC EQUATIONS

One way to solve the equation $x^2 + bx + c = 0$ is to find the x-intercepts of $y = x^2 + bx + c$. You can use a graphing calculator to find an approximate answer that way. For a precise answer, you can use what you learned in the previous section about how to find the x-intercepts from the vertex.

Example: Solve $x^2 + 4x + 1 = 0$.

The solutions to the equation are the x-intercepts of $y = x^2 + 4x + 1$. We have shown that they are equal to $H - \sqrt{-V}$ and $H + \sqrt{-V}$. So all we have to do is find the values of H and V. There are two ways to do that, outlined as follows:

First method: Find H and V by rewriting the equation $y = x^2 + 4x + 1$ into vertex form. This can be done by completing the square.

$$y = x^2 + 4x + 1 = (\text{a perfect square}) - ?$$
$$y = x^2 + 4x + 1 = (x^2 + 4x + \ldots) - ?$$
$$y = x^2 + 4x + 1 = (x^2 + 4x + 4) - 3$$

11. a. Explain the algebraic steps in the three preceding equations.
b. Write $y = x^2 + 4x + 1$ in vertex form.
c. Give the coordinates of the vertex.

Second method: Find H and V by first remembering that $H = -(b/2)$. In this case, $b = 4$, so $H = -(4/2) = -2$. H is the x-coordinate of the vertex. Since the vertex is on the parabola, we can find its y-coordinate, V, by substituting -2 into the equation.

12. a. Find V. Check that it is the same value you found in problem 11.
b. Now that you have H and V, solve the equation.

13. What are the advantages and the disadvantages of each method? Explain.

For each equation, find H and V for the corresponding function. Then solve the equations. There may be zero, one, or two solutions.

14. $y = x^2 + 6x - 9$ **15.** $y = x^2 - 6x + 9$

16. $y = x^2 - 6x - 9$ **17.** $y = x^2 + 6x + 12$

18. How does the value of V for the corresponding function affect the number of solutions? Explain.

QUADRATIC EQUATIONS CHECKPOINT

As of now you know five methods to solve quadratic equations in the form $x^2 + bx + c = 0$. They are listed below.

I. *On Graphing Calculators:* Approximate solutions can be found by looking for the x-intercepts of $y = x^2 + bx + c$.

II. *Factoring* and the zero product property can sometimes be used.

III. *Equal Squares:* First complete the square, then use the equal squares method.

IV. *Using Vertex Form:* Complete the square to get into vertex form, then use the fact that the solutions are equal to $H - \sqrt{-V}$ and $H + \sqrt{-V}$.

V. *Using the Vertex:* Remember that for the function $y = x^2 + bx + c$, $H = -b/2$. Substitute into the equation to find V. Then use the fact that the solutions are $H - \sqrt{-V}$ and $H + \sqrt{-V}$.

Caution: In the next chapter you will learn another way to solve quadratic equations in the more general form $ax^2 + bx + c = 0$. Meanwhile you can solve them by dividing every term by a.

Example: Find an exact solution for:

$$x^2 - 6x + 2 = 0.$$

This does not seem to factor easily, which rules out Method II, and an exact solution is required, which rules out Method I. Luckily, Methods III-V always work on problems of this type. Using Method III:

$$\begin{aligned} x^2 - 6x + 2 &= 0 \\ (x^2 - 6x + 9) - 7 &= 0 \\ (x - 3)^2 - 7 &= 0 \\ (x - 3)^2 &= 7 \end{aligned}$$

So $x - 3 = \sqrt{7}$ or $x - 3 = -\sqrt{7}$, and the solutions are $3 + \sqrt{7}$ and $3 - \sqrt{7}$.

19. Solve the same equation with Method IV or V. Check that you get the same answer.

Solve these equations. Use each of Methods II-V at least once. Give exact answers. The equations may have zero, one, or two solutions.

20. $x^2 - 4x + 2 = 0$

21. $x^2 + 8x - 20 = 0$

22. $x^2 - 14x + 49 = 0$

23. $x^2 - 16x + 17 = 0$

24. $x^2 + 9x = 0$

25. $x^2 + 9 = 0$

THINKING WRITING

13.B Find the Dimensions

You have 40 square feet of artificial turf and 28 feet of fencing. Is it possible to use all your materials to build a rectangular pen?

1. Find the dimensions of a rectangle having area 40 and perimeter 28. (Hint: You may use trial and error, tables, or graphs.)

Problems like this one can be solved using algebra. The first step is to write some equations.

$$\begin{cases} LW = 40 \\ 2L + 2W = 28 \end{cases}$$

2. Explain how these equations express the given conditions for the pen.
3. Divide all the terms in the second equation by two, to make it simpler.
4. Use algebra to show how the equations can be combined into one of the following equations having just one variable:
   a. $L(14 - L) = 40$, or
   b. $L + \frac{40}{L} = 14$
5. Explain the following steps to transform the equation in problem 4b:

$$L + \frac{40}{L} = 14$$
$$L^2 + 40 = 14L$$
$$L^2 - 14L + 40 = 0$$

6. a. Use algebra to transform the equation in problem 4a into the same equation.
   b. Solve the equation.
7. a. The perimeter of a rectangle is 50. Write the area in terms of the length.
   b. The area of a rectangle is 60. Write the perimeter in terms of the width.

For each problem, 8-11, find the dimensions of the rectangle. Show your work and explain your method. Include a sketch labeled with the variables you use.

8. A rectangle has area 180 and perimeter 64.
9. A rectangle has area 126. The length is 25 more than the width.
10. A rectangle has perimeter 35, and its length is 4 times its width.
11. A rectangle has area 25, and its length is 4 times its width.
12. **Report** Hyru has 40 square feet of artificial turf. Valerie has 40 feet of fencing. They decide to use all their materials to build a rectangular pen. Write them a letter explaining as many methods as possible for finding appropriate dimensions for such a pen.

Essential Ideas

PERIMETER AND AREA

1. A rectangle has width $2x + 5$ and length $3x + 1$. What is the area, when the perimeter is 30?

2. The width of a rectangle is five less than the length. Write a formula for:
   a. the length in terms of the width;
   b. the width in terms of the length;
   c. the area in terms of the width;
   d. the perimeter in terms of the length.

3. The perimeter of a rectangle is 50.
   a. Find the dimensions that will give an area of 46.
   b. Find the dimensions that will give the largest possible area.

4. The circumference of a circle is 50. What is the area? (Hint: First find the radius.) Is it bigger or smaller than the area of the largest possible rectangle having perimeter 50?

5.
   a. Find the dimensions and the area of the largest possible rectangle that can be made with P feet of fencing.
   b. Find the area of the circle that is surrounded by P feet of fencing. (Hint: Start by expressing the radius in terms of P.)
   c. Which has greater area, the rectangle or the circle? Explain.

FARES

6. A bus company takes people from a small town to and from a large city where they work. The fare is $4.00 per day, round trip. The company wants to raise its fare and has done a survey to find out if this will cause people to stop riding the bus. They estimate that for every 50 cents that they raise the fare, they will lose approximately 1000 customers. They now have 14,000 customers. Do you think they should raise their fare? If so, by how much? Explain.

7. A spaceship company charges its customers a basic fare of $50 million per light year for trips outside the solar system. However, to encourage long trips, it reduces the fare by $1 million for every light year a customer travels. For example, if a tourist travels five light years, her fare is reduced by $5 million. Her cost will be $45 million per light year for five light years, or $225 million. What is the most a person could ever pay for a trip on this spaceship? Explain.

PARABOLAS AND INTERCEPTS

8. Which graphs have the same x-intercepts? Explain.
   a. $y = x(8 - x)$ b. $y = 2x(8 - x)$
   c. $y = x(2 - x)$ d. $y = x(8 - 2x)$
   e. $y = 3x(8 - 4x)$ f. $y = x(16 - 2x)$

9. Graph the following three functions on the same axes. Label x-intercepts, y-intercept, and the vertex of each parabola.
   a. $y = x(25 - 2x)$ b. $y = x(25 - x)$
   c. $y = 2x(25 - x)$

10. Pick one of the three functions in problem 9 and describe a real situation that would lead to the function. Tell what the variables represent. Make up at least two questions about the real situation that could be answered by looking at the graph you made in problem 9.

11. Write the equation of a parabola having x-intercepts at:
   a. (0, 0) and (2, 0); b. (-4, 0) and (0, 0);
   c. (-4, 0) and (1, 0).

12. Compare the graphs of $y = 4x(x - 1)$, $y = 2x(2x - 2)$, and $y = x(4x - 4)$. Explain what you observe.

13. a. Find the equation of a parabola that has no x-intercepts.
   b. Find the equation of a parabola that has only one x-intercept.
   c. Find the equation of a graph that has three x-intercepts.

14. How many x-intercepts? Explain.
   a. $y = 2x + 1$ b. $y = x(4 - x)$
   c. $y = x^2 + 1$ d. $y = 3(x + 1)^2$

15. How many x-intercepts? Explain.
   a. $y = 8x - x^2$ b. $y = x^2 - x + 2$
   c. $y = 2x^2 + 12x + 18$

16. How many x-intercepts? Explain.
   a. $y = a(x - H)^2$ b. $y = a(x - H)^2 + 3$
   c. $y = a(x - H)^2 - 3$

THE VERTEX

17. a. Write the equation of any parabola that crosses the x-axis at (2, 0) and (4, 0).
   b. Write the equation of any other parabola that crosses the x-axis at these two points.
   c. Find the coordinates of the vertices of both parabolas. Compare them. What is the same? What is different?

18. a. Write the equation of any parabola that crosses the x-axis at (0, 0) and (3, 0).
   b. Write the equation of a parabola that crosses the x-axis at (0, 0) and (3, 0) and has 9 as the y-coordinate of its vertex.
   c. Find an equation of any other parabola that has 9 as the y-coordinate of its vertex.
   d. Compare the three equations. What is the same? What is different?

19. Write three equivalent equations for the parabola that crosses the x-axis at (2, 0) and (0, 0) and has 6 as the y-coordinate of its vertex.

20. Find the equation of a parabola having:
   a. intercepts: (6, 0), (-2, 0), (0, 4);
   b. vertex (-1, -4); one intercept at (1, 0);
   c. vertex (-2, 0); one intercept at (0, 2).

21. Find the coordinates of the vertex of the graph of:
   a. $y = -2(x - 5)(x + 8)$;
   b. $y = (x + 3)^2 - 6$;
   c. $y = x^2 + 4x - 7$.

22. Find the equation of a parabola that has a vertex having the following coordinates:
   a. (2, 8) b. (8, 64)

23.
   a. Write the equation of a parabola that has x-intercepts $(p, 0)$ and $(-r, 0)$. How can you check that your answer is correct?
   b. What are the coordinates of the vertex?

QUADRATIC EQUATIONS

24. Solve.
   a. $(x - 8)^2 + 6 = 0$ b. $(x - 8)^2 - 6 = 0$
   c. $(x + 8)^2 + 6 = 0$ d. $(x + 8)^2 - 6 = 0$

Solve.

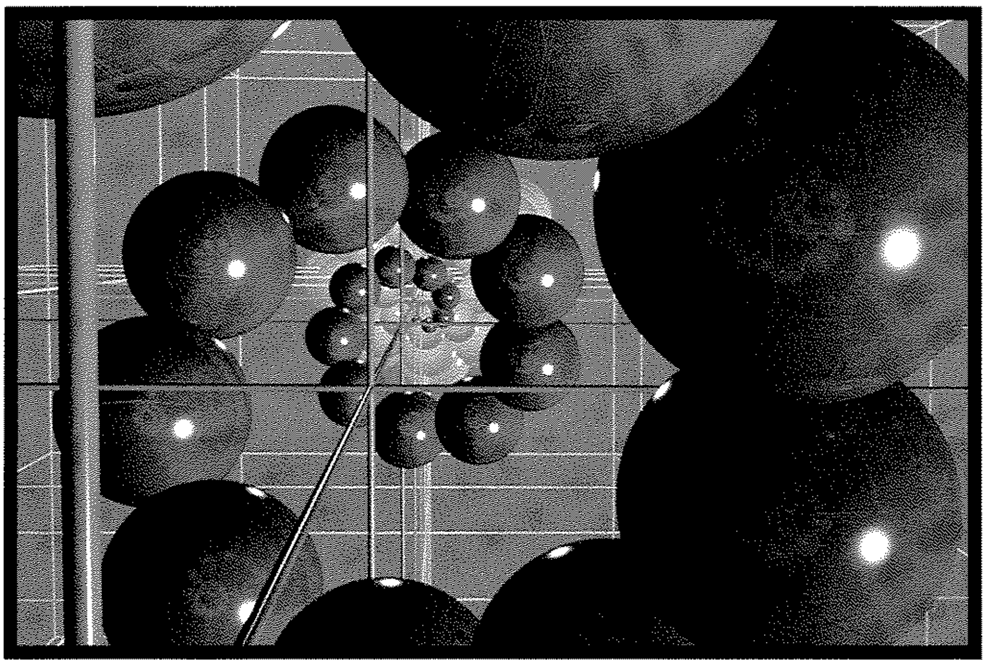
25. $x^2 - 6 = 0$ **26.** $x^2 - 6x = 0$

27. $x^2 - 6x = -9$ **28.** $x^2 + 6x = -9$

29. $x^2 + 6x - 4 = 0$ **30.** $-4x + 2 = -x^2$

31. $-x^2 = 8x + 7$ **32.** $8x - x^2 = 7$

CHAPTER 14



A futuristic spiral

Coming in this chapter:

Exploration Make a paper rectangle that is similar to the smaller rectangle obtained by the following method:

a. cutting the original rectangle in two equal parts;
b. cutting off a square from the original rectangle.

What is the exact ratio of length to width for each of your rectangles?